Part I: Standing Waves

Chapter 176. Standing Waves

In some ways, standing waves are very similar to traveling waves. But in others, they seem to break the usual wave behavior. For example, they don't transmit energy; indeed, they don't transmit anything. They can't reflect or diffract, because they don't travel through the medium. And they have features that are never seen in traveling waves. But we will see in this Part that they are so closely related to traveling waves, it really does make sense to include both of them in a single category of waves.

One reason to learn about standing waves is that sound can form standing waves. But standing waves also have another important connection to sound: they occur in and on many musical instruments. Standing waves are especially likely to occur when the medium is of limited size, so that strings and tubes and drumheads are just the right places for them to occur.

Most of the variety of characteristics described in Chapter 114 apply to standing waves as well as traveling waves. For instance, there can be standing waves of all three dimensionalities, although this book has only a little about two-dimensional and nothing about three-dimensional standing waves. There can also be a variety of ways that the pieces of the medium move. However, in Chapter 114 those motions were categorized relative to the direction of wave travel. Standing waves do not have that reference, so a direct correspondence to traveling wave types is not always clear. For us, two observations will suffice. First, if the density of the medium is varied by the standing wave, then it certainly qualifies as a compression wave. Second, in a one-dimensional medium, which will demand most of the attention here, **transverse** and **longitudinal** waves can be defined relative to the medium.

Chapter 114 carefully defines the extent of a traveling wave, but this is rarely of interest for standing waves. Standing waves most often occur in a medium of limited size, in which case they usually fill the entire medium. The **duration** of a standing wave is then the same thing as how long the wave persists, but unlike with traveling waves that has nothing to do with how large the wave is.

Although standing waves can have many different shapes, sinusoidal shapes are special, just as is the case with traveling waves. Chapter 186 discusses other shapes, but to understand them it is important to first consider the sinusoids.

Figure 176.1 shows a sinusoidal transverse standing wave on a string with two fixed ends. First, focus only on the solid black curve. This is a **snapshot graph,** a graph of the disturbance of the medium versus position at some specific moment in time. Recall that although a snapshot graph sometimes looks like a photograph of the wave, the displacement axis scale is often magnified compared to the axis along the medium.

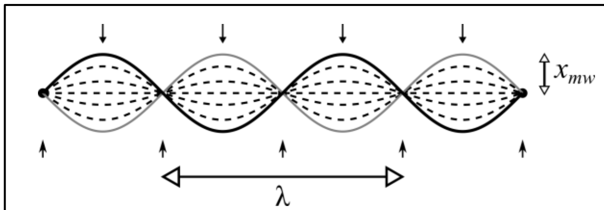


Figure 176.1

Sinusoidal transverse standing wave on a string. Nodes are indicated by arrows along bottom, antinodes are indicated by arrows along top.

To illustrate the motion of the standing wave, the figure shows snapshots from seven different times, all on top of each other. Although this is suggestive of the motion in a standing wave, drawing so many snapshots quickly becomes tiresome. On the other hand, drawing just a single snapshot (for instance, just the solid black curve) produces a graph that is indistinguishable from a traveling wave. Whenever possible, this book will show standing waves by drawing snapshots of the two extremes of motion only. In Figure 176.1 those are the solid black and solid gray lines. You should keep in mind that the wave is vibrating through all the intermediate positions as well.

The motion of a standing wave repeats in time. For instance, starting when the string has the shape of the solid black line, the string will get straighter, and then bend to have the shape of the solid gray line, then return through straight and back to the solid black line. Thus, the ideas of **period** and **frequency** from Chapter 10 apply here. Notice that every small piece of the medium is also moving periodically, with the same period and frequency.

Special features appearing in many standing waves are **nodes,** places where the medium is never disturbed at all. These are marked in Figure 176.1 by the small arrows along the bottom. In contrast, traveling waves *never* have nodes! A snapshot of a traveling wave will have places where the disturbance is zero at that moment, but as the wave moves those positions get disturbed. Notice that the ends of this string are included as nodes, even though they are somewhat special because they are required to be nodes by the fact that the string's ends are fixed in place.

Nodes contrast with **antinodes,** places where the medium is disturbed the most, which are marked in Figure 176.1 by small arrows along the top. Notice that an antinode is a specific position in the medium, not a specific displacement of that position. Figure 176.1 has only four antinodes, not eight. Although the string vibrates maximally at the antinodes, the antinodes themselves do not move at all.

For a sinusoidal standing wave, each football-shaped part between neighboring nodes is called a **loop.** At first glance, it might look as if the standing wave in Figure 176.1 is a series of loops, all the same. However, looking closely we see that the loops differ in whether the top or bottom has the solid black line. This means that for a single snapshot, the shape of this standing wave repeats every two loops. The spatial length of one cycle is called the **wavelength** of the wave, symbolized by the Greek letter λ as shown in Figure 176.1. While the base unit for wavelength is the meter (just like any other length), it is often helpful to consider wavelength to have the unit $\mathrm{m/cycle}$. This is similar to units for period; see Chapter 10. Avoid confusion with the terms: the wavelength is *not* the same as the length of the wave.

For a sinusoidal standing wave, each small piece of the medium is oscillating between two extremes. We don't yet have a particular reason to expect that motion to be SHM (other than the fact that sinusoids seem to pop up everywhere), but that does turn out to be the case. Thus, each position along the string has an amplitude (and peak-to-peak amplitude) for its motion. On the other hand, the **amplitude of the standing wave** as a whole is equal to the largest of these oscillation amplitudes, the amplitude at an antinode. This is indicated by x_{mw} in Figure 176.1, to distinguish it from the regular oscillation amplitude x_m.

Chapter 177. Countermoving Interference

The connection between traveling waves and standing waves is revealed by investigating the superposition of two countermoving sinusoidal traveling waves of the same amplitude and frequency. Superposition of two things with the same frequency has the special name interference. (Chapter 39 gives the same name to the superposition of two vibrations with the same frequency.) Since the two waves are traveling through the same medium, they must have the same wave speed. Since they have the same frequency, then by Eq. 121.2 they must also have the same wavelength.

The result is easier to see in an animation, but this book will do the best it can with static pictures. Figure 177.1 shows snapshot graphs of two such waves (the dashed lines) at eight sequential times. You could think of these as transverse string waves, although the graphs work as any type of one-dimensional wave. After time (h), the movie would start over again with time (a). Thus, the full figure shows one period of a repeating pattern. The thin arrows show how specific wave crests, one on each traveling wave, advance through time.

The solid line in Figure 177.1 shows the superposition of the two snapshots at each time. It turns out that the superposition of equal-wavelength sinusoids is always another sinusoid.

Looking at how the solid line changes with time reveals that these countermoving traveling waves have produced a sinusoidal standing wave! More specifically notice that at the nodes of the standing wave, the two traveling waves are always out of phase, with equal magnitude displacements of opposite sign. At the antinodes of the standing wave, the two traveling waves are always in phase, which is easier to see because it means that the curves of the traveling waves always cross at that point.

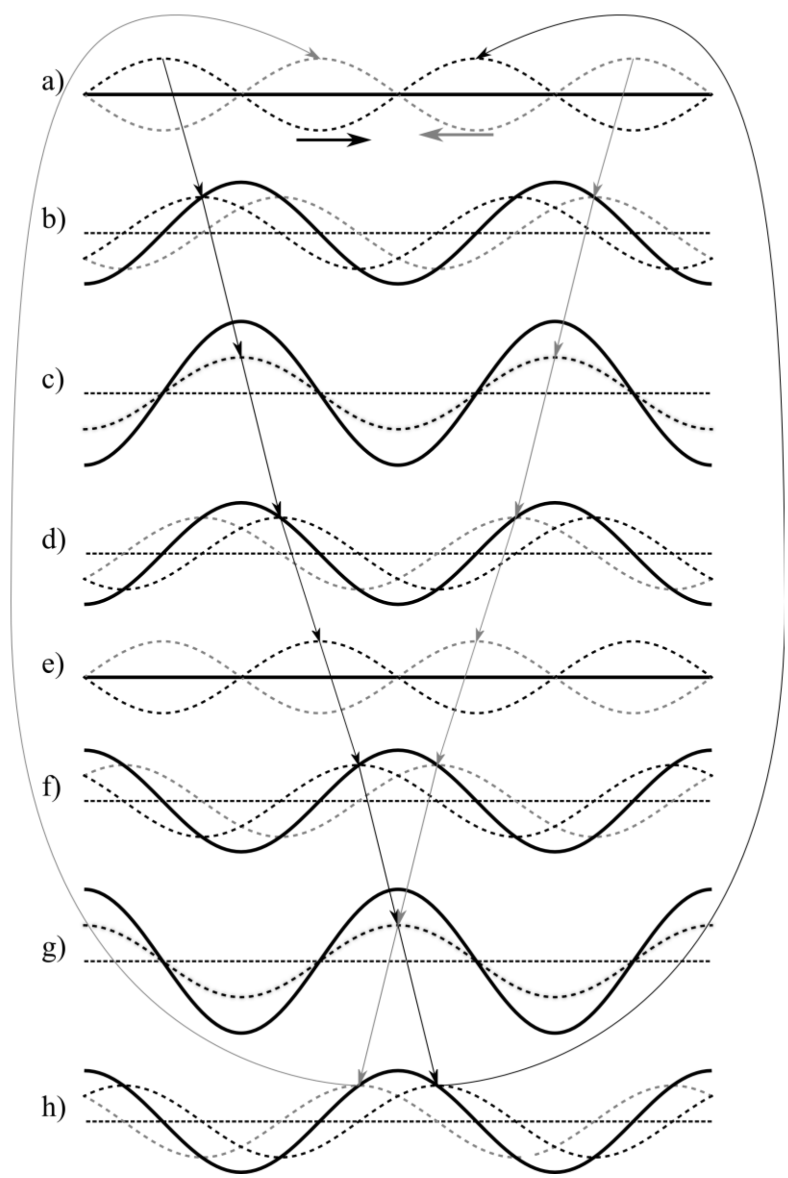


Figure 177.1

Two countermoving sinusoidal waves (dashed black and gray) and their superposition (solid) at eight times through one period of the repetition. Diagonal arrows trace the motion of each wave from one snapshot to the next.

The wavelength of the standing wave is the same as the wavelengths of the traveling waves. In Figure 177.1, the wavelength is half the width of the figure. And the periods of all three are also the same, with the figure showing exactly one repeat of the motion. This means that for standing waves, we can still use

$$s = f\lambda \quad , \tag{177.1}$$

even though for a standing wave nothing appears to be traveling at the speed s. This speed is the rate at which traveling waves would move if they were present alone, so it can be found as usual for traveling waves, with whatever formula or table of information is appropriate for the medium.

We know from Chapter 122 that one traveling sinusoidal wave would cause a particular piece of the medium to move in simple harmonic motion. For the current example, the motion of a particular piece is the superposition of two such motions. Just as the superposition of two snapshots of sinusoids with the same wavelength creates another sinusoid, it is also true that the superposition of two sinusoidal vibrations with the same period creates another SHM. (You may have already seen that in Chapter 39.) This all proves that in a sinusoidal standing wave, each piece of the medium oscillates with SHM. Except, of course, the nodes, which don't move at all—or is that SHM. with zero amplitude?

Chapter 178. Pairs from Reflection

Why, you might ask, would there be two countermoving waves that just happen to have exactly the same frequency? After all, there is no restriction on what frequencies the traveling waves can have. So, while Chapter 177 makes for a nice mathematical relationship, would it ever really happen?

The answer is absolutely yes, because of reflection. When a traveling wave reflects off the boundary of a medium, the reflected wave can automatically provide a countermoving wave of exactly the same frequency as the incident wave. From one point of view, this is why standing waves are closely associated with media of limited size. Limited size means plenty of ends or edges to reflect off

In fact, we can relate the different sorts of reflection behavior, from Chapter 146, to corresponding parts of standing waves.

If the end of the medium is fixed, then reflections are inverted. This means that whatever the displacement of the incident wave at the boundary, the displacement of the reflected wave at the boundary has the same magnitude and opposite sign. When these two are added together in superposition, they add to zero, so that the end is a node. Of course, if the end is fixed, we knew it had to be a node, so there are no surprises here. But it is gratifying that the inversion-on-reflection rule is consistent with the end being fixed.

A more illuminating case is the other extreme, a perfectly free end. In Figure 177.1, we can consider the right side of the figure to be a free end, with the black dashed traveling wave incident on it, and the gray dashed traveling wave being reflected from it. Because the reflection is upright, at the free end the incident and reflected waves have exactly the same displacement. (The two waves are not identical, because the reflected wave is reversed left-to-right.) As is apparent from the figure, the result of superposition is that the standing wave has an antinode at the free end.

Chapter 179. Normal Modes with Fixed Ends

179a. Loop Lengths

This chapter will apply to standing waves the relationship Eq. 121.2, $s = f\lambda$, where f is frequency, λ is wavelength, and s is the wave speed. If you have read Chapter 177 then you have seen why this equation is applicable to standing waves even though nothing about the standing wave appears to be moving at a speed s. Otherwise, you can just take it on faith. We will only consider media that are uniform all along their length, so that s is the same everywhere.

The first consequence of that equation is that if a standing wave is to have alternating nodes and loops as in Figure 176.1, then the loops must all be the same length. Each loop is oscillating up and down with the same frequency. From that, Eq. 121.2 implies that the wavelength is the same everywhere, which means that the loops are all the same length.

Of course, drawing a standing wave with different sized loops is already not a sinusoid, since sinusoids have humps and troughs of equal size. Eq. 121.2 tells us this is closely connected to ensuring that the whole motion has the same frequency. If you need to draw a standing wave, take care to have your nodes and antinodes evenly spaced.

179b. Two Fixed Ends

Consider a 1D medium of length L with two fixed ends, for instance a guitar string. This limits the possible standing waves, because the ends must be nodes. Combine this with the requirement for all loops to have the same length, and the only allowed sinusoidal standing wave shapes are those shown in Table 179.1. They are listed in order of the number of loops present, which in principle has no upper limit. These are called the **normal modes,** or just the **modes,** of vibration for the string. (The word "normal" here does not mean ordinary. Normal is a mathematical term meaning something similar to perpendicular, although describing in what sense these things are perpendicular is beyond the scope of this book.) Pay particular attention to the words "mode" and "node," which mean completely different things, while they sound annoyingly similar.

Table 179.1 lists some characteristics for these standing waves, including expressions in the bottom row that work for any of the modes. The key number here is the number of loops n. For some reason, many students find it attractive to count the number of nodes; while that is certainly closely related to the number of loops, it does not produce the simplest expressions.

The pattern of frequencies in the rightmost column can be broken down as follows. There is a lowest frequency called the **fundamental frequency**

$$f_1 = \frac{s}{2L} \quad . \tag{179.1}$$

The corresponding mode is called the **fundamental mode,** and both the frequency and mode are sometimes just referred to as the **fundamental.**

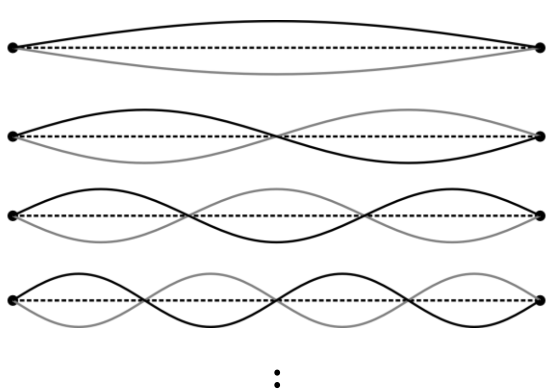
All the other frequencies are positive integer multiples of that lowest frequency, which is called a **harmonic relationship.** This can be expressed algebraically by the equation

$$f_n = nf_1 \quad , \quad n = \text{positive integer.} \tag{179.2}$$

These modes and frequencies are called **harmonics.** The **mode number** n is also called the **harmonic number,** and in this case is equal to the number of loops.

Table 179.1

Possible modes for sinusoidal standing wave on string with two fixed ends.



# loops	loop length	λ	$f = \frac{s}{\lambda}$
1	L	$\frac{2L}{1}$	$\frac{s}{2L}$
2	$\frac{L}{2}$	$\frac{2L}{2}$	$2\frac{s}{2L}$
3	$\frac{L}{3}$	$\frac{2L}{3}$	$3\frac{s}{2L}$
4	$\frac{L}{4}$	$\frac{2L}{4}$	$4\frac{s}{2L}$
n	$\frac{L}{n}$	$\frac{2L}{n}$	$n\frac{s}{2L}$

If you have read Section 44a, you may recognize Eq. 179.2 as essentially identical to Eq. 44.1. The relationships and jargon here are just the same as for the partials of a harmonic spectrum. Mode number here plays a role very similar to partial number in Section 44a.

Chapter 180. Mersenne's Laws

For the specific case of the vibration of a string, Eq. 179.1 can be combined with Eq. 119.1 for the speed of a wave on a string. The result captures all the factors that influence the pitch of a string on a musical instrument such as a guitar or harp,

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}} \quad . \tag{180.1}$$

The three relationships included in Eq. 180.1, relating frequency to string length, tension, and linear mass density, are called **Mersenne's Laws.** None of those relationships are direct proportions. Although Eq. 180.1 is very useful in the specific case of vibrating strings, it is probably a better choice to learn Eqs. 119.1 and 179.1 separately. Each one of those equations has a much wider range of application than just standing waves on a string. You can easily reassemble them to create Eq. 180.1 when the occasion arises.

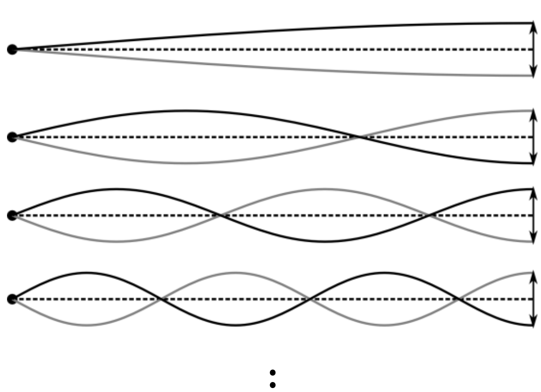
Chapter 181. Normal Modes with Opposite Ends

Consider a 1D medium of length L with one fixed end and one free end. It is unlikely that you will ever see this physically realized for a transverse string wave, because making the end of a string truly free is very difficult. But Chapter 183 shows excellent examples of this case for longitudinal waves.

The fixed end must once again be a node of the standing wave. The free end must be an antinode of the standing wave (see Chapter 178 for the reason). These facts combine with the "all loops the same size" rule to limit the possible standing waves to those shown in Table 181.1. These **normal modes** are listed in order of number of loops. Because of the different ends, there is always an extra half loop.

Table 181.1

Possible modes for sinusoidal standing wave on string with one fixed and one free end.



# loops	loop length	λ	$f = \frac{s}{\lambda}$
$\frac{1}{2}$	$2L$	$\frac{2}{1}2L$	$\frac{1}{2}\frac{s}{2L}$
$1\frac{1}{2} = \frac{3}{2}$	$\frac{2}{3}L$	$\frac{2}{3}2L$	$\frac{3}{2}\frac{s}{2L}$
$2\frac{1}{2} = \frac{5}{2}$	$\frac{2}{5}L$	$\frac{2}{5}2L$	$\frac{5}{2}\frac{s}{2L}$
$3\frac{1}{2} = \frac{7}{2}$	$\frac{2}{7}L$	$\frac{2}{7}2L$	$\frac{7}{2}\frac{s}{2L}$
$m = \frac{\text{odd}}{2}$	$\frac{L}{m}$	$\frac{2L}{m}$	$m\frac{s}{2L}$

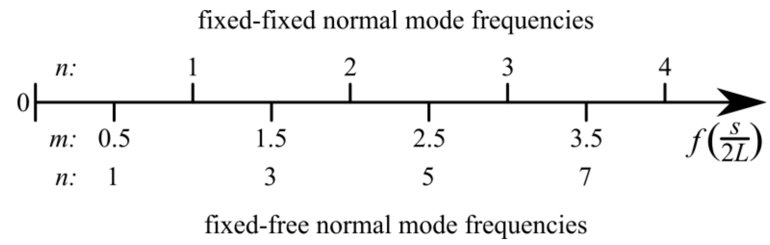


Figure 181.1

For a specific length of the medium, frequencies of normal modes as multiples of $s/2L$. Numbers on the top scale are values for n from Section 179b, and numbers on the bottom scale are values for m and n from Chapter 181.

The general expressions in the last row are formatted so that they turn out to be identical to the expressions in Table 179.1 for two fixed ends. The only difference is that the number of loops is now the variable m, which must be a positive **half integer,**

$$f = m\frac{s}{2L} \quad , \quad m = \text{positive half integer.} \tag{181.1}$$

The term half integer refers to any integer plus one half. (Notice that dividing an integer in half does not necessarily give a half integer. Dividing $(\text{odd})/2$ yields a half integer, but dividing $(\text{even})/2$ yields another integer.)

The **fundamental frequency** is different from the two-fixed-ends case, since the smallest m value gives

$$f_1 = \frac{1}{2}\frac{s}{2L} \quad . \tag{181.2}$$

The frequencies are again **harmonic**, meaning they are whole number multiples of a lowest frequency. By definition, the **harmonic number** of a mode is the ratio of its frequency to the fundamental frequency. So in this case, the **harmonic number** n is not equal to m, but is given by

$$f_n = nf_1 \quad , \tag{181.3}$$

$$n = \frac{f_n}{f_1} = \frac{m\frac{s}{2L}}{\frac{1}{2}\frac{s}{2L}} = 2m \quad . \tag{181.4}$$

Since all m values are half integers, the harmonic numbers n are only odd numbers, so that the allowed frequencies are only the *odd harmonics*. These frequencies and the normal mode frequencies from Section 179b are compared in Figure 181.1 for a single string length.

Sometimes, the modes in Table 181.1 are referred to by which number row they are in. This is called the **mode number,** and it is equal to $m + \frac{1}{2}$. For example, for a medium with fixed end and free end, the 4th mode (with the 4th lowest frequency), has $m = 4 - \frac{1}{2} = \frac{7}{2}$, so it has hamonic number $n = 2m = 7$ and a frequency

$$f_7 = 7f_1 = \frac{7}{2}\frac{s}{2L} \quad . \tag{181.5}$$

If you have read Section 44b, then you may recall another situation calling for only odd harmonics: the partials of a square wave. This is ***not*** to say that these standing waves somehow create a square wave. The situations are quite different; recall that a "square wave" isn't even really a wave. But it is a nice point of similarity, perhaps useful in memorization.

Chapter 182. Other Normal Modes

Very often free ends and fixed ends provide excellent models for real standing waves. This book primarily focuses on those cases. But examples where the ends of the medium are in between, partially constrained, are easy to find. In such cases, any given medium still has normal modes, which are defined by the fact that they vibrate at specific frequencies. Those modes will exhibit nodes and/or antinodes (equally spaced, if there are multiple ones). But the ends of the normal modes will be somewhere between a node and an antinode. The frequencies may not exhibit the mathematical relationships described here.

In fact, strictly speaking, medium ends are never perfectly free or fixed. The ends of the medium containing the standing wave always contact something, and that contact can carry away some of the vibration energy. If you have read Chapter 149, this can be understood in terms of Figure 149.1. Perfectly free or fixed boundaries, the far left and right edges of that figure, reflect 100% of any wave, and would contain the energy of a standing wave. Conversely, if energy can leave the standing wave, that means that the boundaries not at the very edges of that figure, and thus are neither perfectly fixed nor perfectly free.

It is not entirely clear how best to integrate these situations into Eqs. 179.1, 179.2, and 181.1. When the deviations from fixed or free are small, it's handy to think of the end of the standing wave (either a node or an antinode) as being slightly displaced from the physical end of the medium. The length of this standing wave is then called the **effective length** of the medium. But more generally, conditions at the ends of the medium are the reason that n and m can only take specific values. From that perspective, it makes more sense to describe partially constrained ends by allowing n to have different values. The allowed values of n will always be discrete, spaced apart by approximately 1, but they need not be restricted to integers or half-integers.

Chapter 183. Compression Standing Waves

183a. 1D Standing Sound Waves

A tube filled with a fluid, such as air, provides a way to have one-dimensional longitudinal standing waves. The two ends of the tube are the ends of the medium. Each end could partially connect to the fluid outside of the tube, by having differently sized holes. But in this chapter, we will consider only the extreme options: the end being capped off or **closed,** and the tube walls simply ending so that the end is **open** to the surrounding fluid. The two end options give three options for the tube as a whole: an **open tube** has both ends open, a **closed tube** has one end closed and one open, and a **doubly closed tube** has both ends closed. The doubly closed tube is not encountered very often; since its interior is entirely separated from the exterior, it is not easy to use in a musical instrument.

What sorts of standing waves can happen in these tubes? How do they relate to the standing string waves in Chapters 179–181? These waves can be considered either longitudinal or compression waves. The two are synonymous, but the names imply a focus on different characteristics. Either viewpoint is adequate to find the equations for the normal mode frequencies. Comparing the two viewpoints gives a deeper understanding of the standing waves themselves.

183b. Longitudinal Standing Waves

The left and right ends of the tube cross sections in Figure 183.1 are closed and open respectively, forming a closed tube. Focusing on the longitudinal displacements in the wave, at a closed end the displacement of the fluid is essentially fixed at zero. If the fluid were to move away from the closed end, then a gap of nothingness would have to open up, which just doesn't make sense. The restoring force is too strong to allow that.

At an open end, the displacement of the fluid is nearly free. Remember that whether an end is free refers to influences other than the medium restoring force. It's true that if air just inside the open end is displaced

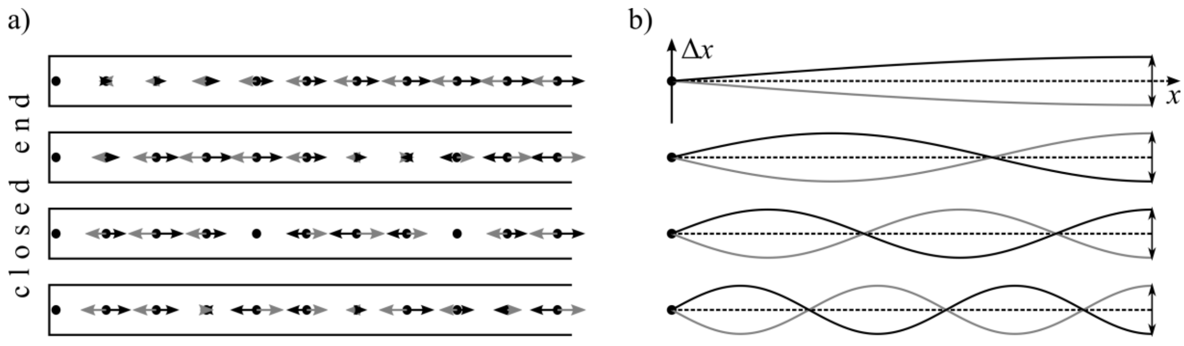


Figure 183.1

Normal modes for longitudinal waves in closed tube. (a) Cross sections of the tubes contain dots at equilibrium positions and arrows to show displacement from them. (b) Displacement versus equilibrium position. The black and gray, both displacement arrows and graphs, represent alternating extremes of the motion.

outward, then the resulting rarefaction in the tube will suck it back in. But that fact doesn't prevent this from being a free end, because that is the restoring force that causes the equilibrium state. The open end is not perfectly free, because in order to displace that air outwards, it needs to push aside the air outside of the tube. Nevertheless, the external air will flow around in response to very small density differences, especially compared to the internal air constrained by the tube walls. So, it is quite a good model to consider an open end to enable free displacement.

A closed tube end is a fixed displacement end.

An open tube end is a free displacement end.

Figure 183.1 shows the normal modes for a closed tube. They correspond to the modes in Table 181.1. In part (a), the displacements are represented in their actual direction at a selection of points throughout the tube. Each arrow shows the maximum displacement for that parcel of air. The black arrows show displacements that all occur at the time of an extreme of wave motion (notice that some are rightward and some are leftward), and the gray arrows show displacements that occur together at the opposite extreme. Clearly, this is not a very practical way to show these standing waves in any great detail.

In Figure 183.1(b) the same modes are shown as graphs of longitudinal displacement versus equilibrium position. These graphs exactly match the graphs in Table 181.1, but here the vertical axis has a different meaning. Keep in mind, in particular, that these graphs *do not* represent something careening from side to side as it hurtles down the tube. The vertical axis of the graph represents horizontal motion, and each curve (e.g., the black or the gray) represents a single moment in time, not a sequence of events.

The same exact discussion from Chapter 181 applies here. So Eqs. 181.1–181.3 apply. A closed tube will have standing wave frequencies that are odd harmonics.

183c. Compression with Different Ends

The left and right ends of the tube cross sections in Figure 183.2 are closed and open respectively, forming a closed tube. Focusing on the compression of the medium, at a closed end, since the interior of the tube is separated from the exterior, and the fluid is free to change density without influence from the outside. The air can crowd up against the closed end, increasing its density, or stretch away from the end, decreasing its density.

At an open end, the fluid is connected to the fluid outside the tube, and that outside fluid has some density that relatively fixed. It is not perfectly fixed. Sound can travel through the outside fluid, which varies its density. But the fluid inside the tube is more able to vary its density because of the containing walls. So it is quite a good model to consider an open end to be a fixed density end. (If you have read Chapter 129, you might prefer to think of this as a fixed pressure end.)

An open tube end is a fixed density (and pressure) end.

A closed tube end is a free density (and pressure) end.

Figure 183.2 shows the normal modes for a closed tube. In part (a), the densities are represented by shades of gray, with darker indicating a denser state. The upper rectangle in each picture shows the density at one extreme of the oscillation, and the lower rectangle shows the opposite extreme. Although the pictures are fairly effective at conveying the variations, they are not very good for showing quantitative values and difficult to draw.

In Figure 183.2(b) the same modes are shown as graphs of density versus position. The open end is fixed and the closed end is free. These graphs match the images in Table 181.1, except that the vertical axis has a different meaning and they have been reversed left-to-right. Keep in mind, in particular, that these graphs *do not* represent something wiggling like a snake in the tube. Each curve (e.g., the black or the gray) represents a single moment in time, not a path of motion.

Reversing the graphs left-to-right does not change the conclusions from Chapter 181, so Eqs. 181.1–181.3 apply. A closed tube will have standing wave frequencies that are odd harmonics.

If you have read Chapter 183, then you probably noticed that this chapter is very similar, but reversed. The reversal is a direct consequence of the relationship between displacement and density in longitudinal waves seen in Chapter 124. In that chapter the extremes of displacement occurred at the same locations as the zeros of density change, and vice versa. In this chapter, the displacement antinodes occur at the density nodes, and vice versa.

But because these are just two different ways to look at the same standing waves, the formulae for the frequencies are the same either way. The closed tube has normal modes with the frequencies given by Eqs. 181.1–181.3 whether you consider the open end to be the free (displacement) end or the fixed (density) end..

183d. Extra: Compression with Similar Ends

The standing wave shapes allowed by a 1D medium with two free ends are not shown in this book; they are left as an exercise for the reader. But, notice that the case of a tube with two open ends can be considered as either having two fixed compression ends, or two free displacement ends. Conversely, a doubly closed tube has those two perspectives reversed. This implies, quite correctly, that the allowed frequencies are exactly the same whether both ends are fixed or both ends are free. You may find it more useful to consider Eqs. 179.1 and 179.2 to be the “ends the same” case, while Eqs. 181.1–181.3 would be considered the “ends opposite” case.

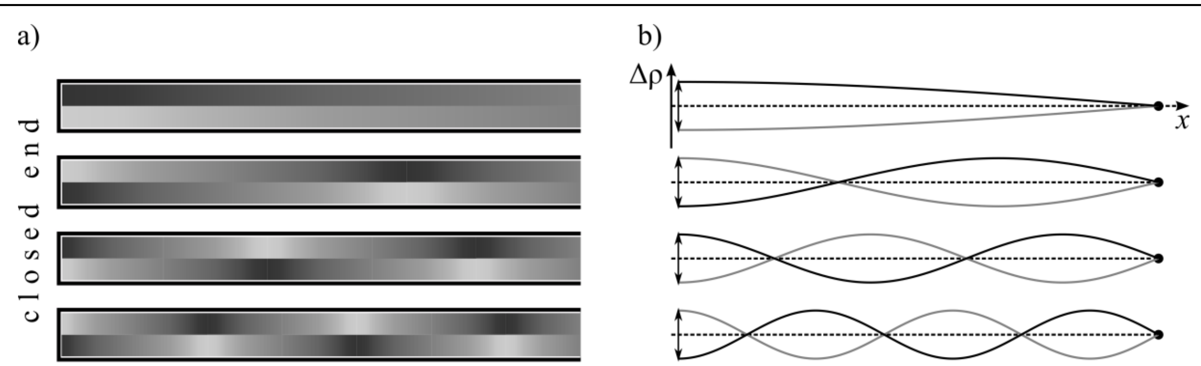


Figure 183.2

Normal modes for compression waves in closed tube. (a) Cross sections of tubes containing gray scale representing density, darker being denser. Top and bottom panels represent alternating extremes. (b) Density extremes versus position.

Chapter 184. End Correction

Chapter 183 offers justifications for modeling an open tube end as either a free displacement end or a fixed density end. However, it doesn't take much experimentation to discover that this model is not quite good enough to tune tubes to specific frequencies. A thought experiment about traveling waves can suggest the reason. Imagine a compression pulse traveling down a tube towards an open end, as in Figure 148.1(c). When it reaches the end, it is easy to imagine that the compression will continue out into the external air a little way, before it expands and disperses. This suggests that the true "end" of the medium containing the wave is actually somewhat outside of the open end.

The physics required to resolve this question is quite advanced, but it confirms that the effective or **acoustic length** L_a of an open pipe is slightly longer than its **physical length** L. When accounting for this, it is the acoustic length that should be used in formulae, such as Eqs. 179.1 and 181.2. The extra length, called the **end correction** ΔL, can be defined for a closed tube by

$$L_a = L + \Delta L \quad . \tag{184.1}$$

An open tube has the same situation at both ends, so the end correction must be added twice.

For a tube of circular cross section that simply ends, as long as the standing wave wavelength λ is long enough compared to the tube radius r, the end correction is well approximated by[60]

$$\Delta L = 0.61r \quad \text{if} \quad \lambda > 8\pi r \quad . \tag{184.2}$$

If the wavelength is shorter than that, the length correction also gets smaller, to the point that if $\lambda = 2r$, the length correction is about $\Delta L \approx 0.3r$. If the wavelength gets much shorter than that, it no longer even makes sense to consider the sound as a 1D wave.

For a tube of a fixed physical length L, higher numbered modes will have shorter wavelengths. The condition in Eq. 184.2 can therefore be expressed in terms of a maximum mode number as

$$\Delta L = 0.61r \quad \text{if} \quad \text{mode number} < L/4\pi r \quad . \tag{184.3}$$

This condition works for both open and closed tubes, although for a closed tube keep in mind that the mode number is different from the harmonic number. Above this maximum mode number, the frequencies predicted from Chapters 179–181 will no longer follow their harmonic series. Or to give a different interpretation, those equations are still accurate but the effective length is changing with mode number.

Chapter 185. Energy in Standing Waves

Standing waves may not transport energy, but they do contain mechanical energy, both potential and kinetic. Exactly where that energy is situated is not the same as with traveling waves (see Chapter 142 and following), since the two wave types move differently.

Potential energy depends only on the shape of the medium, not its motion. So, the conclusions about one-dimensional traveling waves in Chapter 143 apply equally well to standing waves: the potential energy is concentrated near the places where a snapshot of the medium displacement varies along the medium. For sinusoidal standing waves this means that potential energy gets the biggest at the nodes of displacement.

This is equally accurate for transverse and longitudinal waves. But for longitudinal waves, it is probably easier to think of then as a compression waves. The potential energy is then concentrated where the density change is furthest from zero. For sinusoidal waves, this occurs at the antinodes of compression.

[60] Harold Levine and Julian Schwinger, "On the Radiation of Sound from an Unflanged Circular Pipe," *Physical Review* 73(4) (1948): 383–406.

The total potential energy also changes through the cycle of motion. Total potential energy is greatest when the sinusoidal standing wave is at extremes of disturbance, and potential energy drops to zero twice per cycle as the medium passes through its equilibrium state.

Kinetic energy in standing waves is quite different from the traveling wave case. Sinusoidal standing waves are less confusing than traveling waves. Since displacement nodes never move, they never have kinetic energy. Other points in the medium do move, with the maximum kinetic energy occurring at the displacement antinodes (which are the compression nodes). The total kinetic energy also changes in time, very much like a mass on a spring. It is greatest when the medium passes through its equilibrium state, when each piece of the medium achieves its greatest speed. When the medium reaches extremes of motion, the kinetic energy drops to zero.

The combined picture for sinusoidal standing waves is thus very similar to the case of a mass on a spring. The energy in the wave cycles between potential and kinetic forms, going through two such cycles for every period of the motion. As it does so, it also sloshes back and forth between nodes and antinodes. Which one corresponds to which type of energy depends on whether the wave is described with displacements or compressions.

In the most general sense, this picture can be reconciled with the description in Chapter 177 of how a standing wave can be considered the result of the superposition of two countermoving traveling waves. If each of those underlying traveling waves is transporting energy in opposite directions, then it makes sense that the net effect should be energy that is pretty much stationary. But the details, such as the forms of that energy or the sloshing back and forth, can't be discovered by considering only the energy of the interfering waves.

Chapter 186. Standing Wave Superposition

Chapters 179–182 describe the possible normal modes for sinusoidal standing waves. Standing waves can have other shapes. However, we will find that only sinusoidal standing waves can be described by a single shape that only varies in amplitude. All other standing waves change shape as they vibrate. The further one pursues this question, the less clear the distinction between standing and traveling waves becomes.

How can we organize all possible standing wave shapes? The situation is similar to that in Chapter 42, when considering all the possible complex vibrations. And the same solution applies, a slightly modified version of **Fourier's theorem**.

> Any standing wave can be built up by superposition from sinusoidal standing waves. You never need more than one sinusoidal standing wave at each frequency (and wavelength). In a medium with boundaries, such that the only allowed sinusoidal waves make a set of normal modes, any standing wave is a superposition of normal modes.

When the normal mode frequencies are harmonically related, this means that the frequency of the combined standing wave must also be a harmonic of the fundamental mode frequency. Except in relatively special combinations, the frequency of the combined standing wave will equal the fundamental. To put that another way, if you make one of these 1D mediums vibrate, it will almost certainly vibrate at the fundamental frequency, even though higher harmonics may be involved in the vibration.

For a superposition of normal modes, it is their different frequencies that guarantee that the wave shape will change with time, more than just varying amplitude. In fact, the result sometimes takes on characteristics of a traveling wave. Figure 186.1(a) shows one example, made on a fixed-end string from the lowest three normal modes, with appropriate amplitudes. The wave progresses from a solid curve through longer to shorter dashed curves, and then back again. Even with only three components, this wave has a definite sense of a bump bouncing from end to end.

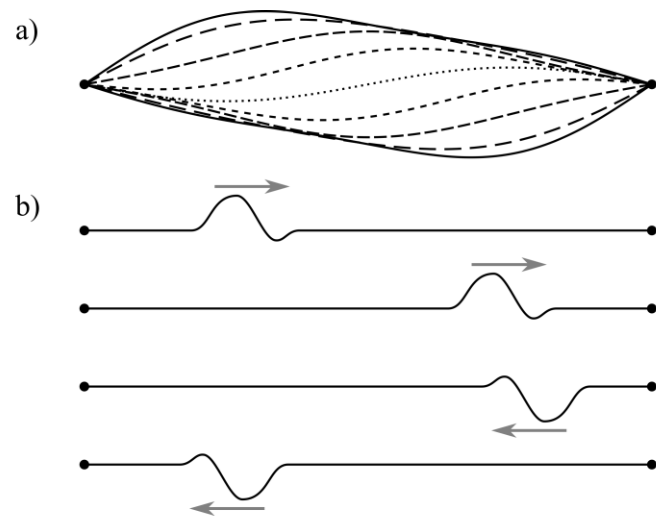


Figure 186.1

Standing waves on a string fixed at both ends with traveling characteristics. (a) Example made from lowest three normal modes. (b) A trapped wave pulse, which would require very many normal modes in superposition.

Even an idealized pulse bouncing between the ends of a finite medium, as in Figure 186.1(b), can be built from the superposition of standing waves (although this example would take very, very many normal modes to produce). Is this still a standing wave? In one sense, the wave energy never leaves the string, but on the other hand it is clearly moving back and forth. The most important lesson here is that the boundary between standing waves and traveling waves is not as clear-cut as one might have expected.

If we are willing to call even some of these standing waves, then nodes and antinodes are not necessary parts of standing waves. Several of the snapshot lines in Figure 186.1(a) have points of zero displacement, but those are not nodes because they do not stay that way forever.

This chapter highlights that when working in physics, it is sometimes possible to reach conclusions even when it is not feasible to fill in all intermediate details. It would be very difficult to determine which normal modes with which amplitudes and phase constants would be required to build Figure 186.1(b) through superposition. Also, that wouldn't be a smart approach, since that wave is more easily understood as a reflecting traveling wave. Nevertheless, Fourier's theorem is so reliable that we know that it could be done, and with that knowledge comes some insight into the nature of standing waves.

Chapter 187. 2D Standing Waves

Standing waves can occur in 2D and 3D media as well as in 1D. All that is needed is pairs of countermoving traveling waves. In this chapter we will only look at some qualitative features of 2D standing waves that occur on thin plates and membranes, without worrying about the traveling waves that underlie them. Membrane refers to something like a drumhead — a thin flexible sheet that is stretched out to provide a restoring force towards a particular shape, usually flat. The restoring force is in some ways similar to that for a stretched string, and for the same reasons the edge of a stretched membrane is nearly always fixed in place. The word plate in this context refers to a thin rigid material, so that the restoring force comes from the rigidity of the material itself. Again, the common shape is close to flat, as for a cymbal or gong. But the same general concepts also apply to other rigid shapes, such as a cup or bell, that a flat plate could deform into.

Two-dimensional waves in all of these media are made from displacements of the medium in a direction perpendicular to the surface, so they are transverse waves. As with 1D standing waves, these media have special standing waves, with the characteristic that they fill the medium so that all pieces are oscillating with the same frequency. These are the normal modes. These media can have more complicated standing waves as well, but those can be described as superpositions of multiple normal modes.

However, the frequencies of these normal modes are *not* likely to be harmonically related, even for simply shaped objects. This is the reason that musical instruments based on these shapes generally do not create a strong sense of pitch. Those which do have a particular pitch either have one mode that dominates the others in amplitude, or they have been carefully shaped so that the frequencies of many of the modes are nearly harmonically related.

Like 1D normal modes, there can be some positions on the plate or membrane where the vibration amplitude is zero. But unlike in one dimension, this never occurs at only one isolated point. Instead, there are **nodal lines** along the medium, with zero motion at every point on the line. In the late 1700s, Ernst Chladni became famous for a technique to reveal these nodal lines on rigid plates,[61] illustrated in Figure 187.1(a). The plate is secured in a horizontal position, usually only at one point so that the rest of it can vibrate. Then the plate is made to vibrate. (In Chladni's case, he drew a violin bow across the edge.) Fine sand sprinkled on the plate collects along the nodal lines. The patterns formed can be quite intricate; an internet search will find numerous pictures and videos.

a)

b)

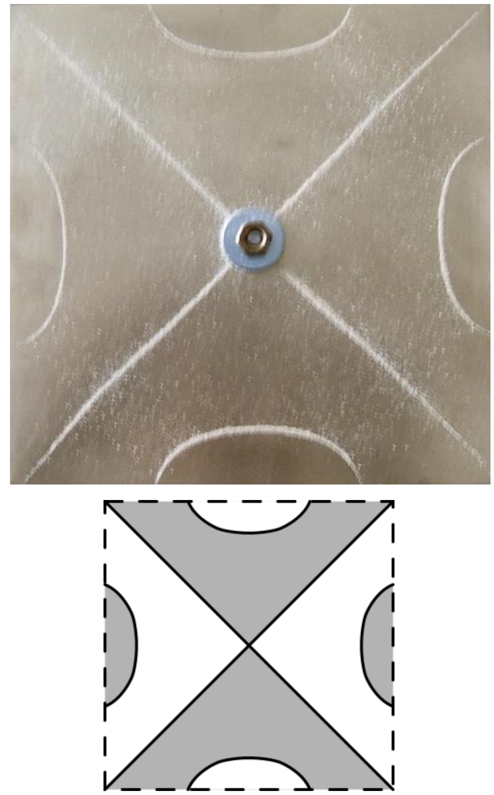


Figure 187.1

Chladni pattern revealing nodal lines for one vibrational mode of a square plate. (a) Reprinted from Elmar Bergeler, *Wikimedia Commons,* 2011, https://commons.wikimedia.org/wiki/File:Chladni_pattern_2.jpg (June 2015) / CC BY-SA 3.0. (b) Schematic: lines are nodal lines, shading shows phase.

In the regions between the nodal lines, there are places where the vibration amplitude is maximum. These **antinodes** are nearly always points, not lines.

The patterns formed by the nodal lines for any specific case are beyond the scope of this book. However, there are some general rules satisfied by all normal modes which can be easily described.

- The regions on either side of a nodal line, including the antinodes, vibrate out of phase with each other. when one region has a positive displacement, the other region has negative displacement, and vice versa.
- As a consequence, nodal lines can't end in the middle of the plate or membrane. If one did, then the region in which it ended would have to be out of phase with itself, which is impossible. The nodal lines either make complete loops, or they terminate at the edge of the medium.
- Nodal lines can cross, and in fact there can be points from which many nodal lines radiate. The number of lines radiating from such a point must be even, again because of the out-of-phase rule. For example, four nodal lines extend from the center of Figure 187.1 (which you might equally well consider as two lines crossing in the center).

These three rules mean that a nodal line pattern can be colored to show the phase of the regions. In Figure 187.1(b) all the gray regions are in phase with each other and out of phase with all the white regions. Every nodal line has gray on one side and white on the other. Notice that the edge of the plate is not marked with a solid line because it is not a nodal line.

61 Ernst Chladni, *Entdeckungen über die theorie des klanges,* Leipzig: Weidmanns Erben und Reich, 1787.

Three other rules for these patterns do not relate to the phase of vibration.

- If the plate material is uniform, then the vibrating regions between the nodal lines will all be roughly the same size. This is the 2D analog to the loop size rule in Section 179a, and has the same reasoning behind it.
- Wherever the medium is held stationary, at least one nodal line must pass through that position. Thus, the edge of a membrane (but not a plate) is a nodal line, similar to the ends of a 1D string. The center of Figure 187.1(a) is fixed, and therefore on a nodal line (in fact, two for the mode shown). The square plate in Figure 187.1 may have some vibrational modes for which the center is *not* on a nodal line, but they could not be demonstrated with that particular setup.
- Wherever the medium is made to vibrate, by striking or otherwise disturbing, a nodal line cannot pass through that position. In many modern demonstrations of Chladni patterns, the position where the plate is held is also the position that is vibrated. That method does not force the nodal lines to pass through any particular position, but it does prevent them from certain positions. For example, if the center point in Figure 187.1 were the source of vibration, it would not be possible to make the pattern shown in the figure.

This last rule is often used to good advantage by percussion players. By striking their instruments at different points, they excite different modes, and thus produce sounds with different timbres.