building, as things to be laid aside or got rid of as soon as finite lines were found proportional to them. But then these finite exponents are found by the help of fluxions. Whatever therefore is got by such exponents and proportions is to be ascribed to fluxions: which must therefore be previously understood. And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?[3]

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13 MACLAURIN. ON SERIES AND EXTREMES

Berkeley's criticism stung, and during the eighteenth century many attempts were made to place the calculus on a rigorous foundation. For a report on these attempts, as far as Great Britain is concerned, see F. Cajori's works quoted in the introduction to Selection V.12. One of the most distinguished attempts was made by the Edinburgh professor Colin Maclaurin (1698–1746) in his *Treatise of fluxions* (Edinburgh, 1742). Maclaurin started, like Barrow and Newton, from the concepts of space, time, and motion. But Maclaurin's book also contains other contributions. Best known is his introduction of Taylor's series in a way that has remained familiar in elementary textbooks. He gave the method for deciding between a maximum and a minimum by investigating the sign of a higher derivative. Here follow, in the original text, some of the articles of the *Treatise* that contain these contributions. Maclaurin also considered questions of convergence in series. See H. W. Turnbull, *Bi-centenary of the death of Colin Maclaurin* (University Press, Aberdeen, 1951), also "Colin Maclaurin," *American Mathematical Monthly 54* (1947), 318–322, and our Selection III.10, note 3.

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751. The following theorem is likewise of great use in this doctrine. Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. represent invariable coefficients as usual, any of which may be supposed to vanish. When z vanishes, let E be the value of y, and let $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. be then the respective values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. z being supposed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. the law of the continuation of which series is manifest. For since $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = o$,

[3] We may think here of the many arguments involved in the Zeno paradoxes, which also played a role in the eighteenth-century discussions concerning the foundations of the calculus; see Cajori, *History of the conceptions of limits and fluxions*, quoted in the introduction to this selection, and his nine articles, "History of Zeno's arguments on motion," *American Mathematical Monthly 22* (1915).

A is equal to y; but (by the supposition) E is then equal to y; consequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = o$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is supposed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. Let $z = o$, and substituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by supposing $z = o$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. This proposition may be likewise deduced from the binomial theorem. Let BD [Fig. 1], the ordinate of the figure

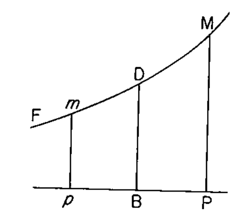

Fig. 1

FDM at B, be equal to E, $BP = z$, $PM = y$, and this series will serve for resolving the value of PM, or y, (some particular cases being excepted, as when any of the coefficients E, $\frac{\dot{E}}{\dot{z}}$, $\frac{\ddot{E}}{\dot{z}^2}$, &c. become infinite) into a series, not only in such cases as were described in the preceding articles, but likewise when the relation of y and z is determined by an affected equation, and in many cases when their relation is determined by a fluxional equation. This theorem was given by Dr. Taylor, *method. increm.* By supposing the fluxion of z to be represented by BP, or $\dot{z} = z$, we have $y = E + \dot{E} + \frac{\ddot{E}}{2} + \frac{\dddot{E}}{6} + \frac{\ddddot{E}}{24} +$ &c. (as was observed in Art. 255)[1] and hence it appears at what rate the fluxion of y of each order contributes to produce the increment or decrement of y, since $y - E = \dot{E} + \frac{\ddot{E}}{2} + \frac{\dddot{E}}{6} + \frac{\ddddot{E}}{24} +$ &c. If Bp be taken on the other side of B equal to BP, then $pm = A - Bz + Cz^2 - Dz^3 +$ &c. $=$ (the same quantities being represented by $\frac{\dot{E}}{\dot{z}}$, $\frac{\ddot{E}}{\dot{z}^2}$, &c., as

[1] Maclaurin's book is divided into two parts. Book I is geometrical, Book II is computational. Our selection is from Book II. Articles 255 and 261 (to which he refers below) deal with the same matter in a geometrical way.

before, or the base being supposed to flow the same way,) $E - \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} - \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} -$ &c. consequently $PM + pm = 2E + \frac{2\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{2\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. . . .

Then, in Arts. 858–861, Maclaurin gives his criterion for maxima and minima.

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858. When the first fluxion of the ordinate vanishes, if at the same time its second fluxion is positive, the ordinate is then a *minimum*, but is a *maximum* if its second fluxion is then negative; that is, it is less in the former, and greater in the latter case than the ordinates from the adjoining parts of that branch of the curve on either side. This follows from what was shewn at great length in *Chap. 9. B.* I, or may appear thus. Let the ordinate $AF = E$, $AP = x$ [Fig. 2],

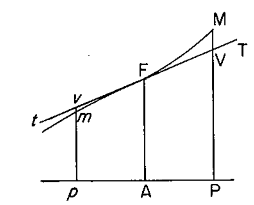

Fig. 2

and the base being supposed to flow uniformly, the ordinate $PM =$ (Art. 751) $E + \frac{\dot{E}x}{\dot{x}} + \frac{\ddot{E}x^2}{2\dot{x}^2} + \frac{\dddot{E}x^3}{6\dot{x}^3} +$ &c. Let Ap be taken on the other side of A equal to AP, then the ordinate $pm = E - \frac{\dot{E}x}{\dot{x}} + \frac{\ddot{E}x^2}{2\dot{x}^2} - \frac{\dddot{E}x^3}{6\dot{x}^3} +$ &c. Suppose now $\dot{E} = o$, then $PM = E * + \frac{\dot{E}x}{\dot{x}} + \frac{\ddot{E}x^2}{2\dot{x}^2}$ &c. and $pm = E * + \frac{\ddot{E}x^2}{2\dot{x}^2} -$ &c. Therefore if the distances AP and Ap be small enough, PM and pm will both exceed the ordinase AF when $\ddot{E}$ is positive; but will be both less than AF if $\ddot{E}$ be negative. But if $\ddot{E}$ vanish as well as $\dot{E}$, and $\dddot{E}$ does not vanish, one of the adjoining ordinates PM or pm shall be greater than AF, and the other less than it; so that in this case the ordinate is neither a *maximum* nor *minimum*. We always suppose the expression of the ordinate to be positive.

859. In general, if the first fluxion of the ordinate, with its fluxions of several subsequent orders, vanish, the ordinate is a *minimum* or *maximum*, when the number of all those fluxions that vanish is 1, 3, 5, or any odd number. The ordinate is a *minimum*, when the fluxion next to those that vanish is positive; but a *maximum* when this fluxion is negative. This appears from Art. 261, or by comparing the values of PM and pm in the last article. But if the number of all the fluxions of the ordinate of the first and subsequent successive orders that vanish be an even number, the ordinate is then neither a *maximum* nor *minimum*.

860. When the fluxion of the ordinate y is supposed equal to nothing, and an equation is thence derived for determining x, if the roots of this equation are all unequal, each gives a value of x that may correspond to a greatest or least ordinate. But if two, or any even number of these roots be equal, the ordinate that corresponds to them is neither a *maximum* nor *minimum*. If an odd number of these roots be equal, there is one *maximum* or *minimum* that corresponds to these roots, and one only. Thus if $\frac{\dot{y}}{\dot{x}} = x^4 + ax^3 + bx^2 + cx + d$, then supposing all the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ to be real, if the four roots are equal there is no ordinate that is a *maximum* or *minimum*; if two or three of the roots only are equal, there are two ordinates that are *maxima* or *minima*; and if all the roots are unequal there are four such ordinates.

861. To give a few examples of the most simple cases. Let $y = a^2x - x^3$, then $\dot{y} = a^2\dot{x} - 3x^2\dot{x}$ and $\ddot{y} = -6x\dot{x}^2$. Suppose $\dot{y} = 0$, and $3x^2 = a^2$ or $x = \frac{a}{\sqrt{3}}$, in which case $\ddot{y} = \frac{-6a\dot{x}^2}{\sqrt{3}}$. Therefore $\ddot{y}$ being negative, y is a *maximum* when $x = \frac{a}{\sqrt{3}}$, and its greatest value is $\frac{2a^3}{3\sqrt{3}}$. If $y = aa + 2bx - xx$, then $\dot{y} = 2b\dot{x} - 2x\dot{x}$, and $\ddot{y} = -2\dot{x}^2$; consequently y is a *maximum* when $2b - 2x = 0$, or $x = b$. If $y = aa - 2bx + xx$ then $\dot{y} = -2b\dot{x} + 2x\dot{x}$, and $\ddot{y} = 2\dot{x}^2$; consequently y is now a *minimum* when $x = b$, if a be greater than b.

Maclaurin also considers the cases in which $\ddot{y}, \dddot{y}, \ddddot{y}, \ldots$ vanish.

14 D'ALEMBERT. ON LIMITS

Among the mathematicians who seriously tried to come to an understanding of the foundations of the calculus (the "metaphysics of the calculus") was Jean LeRond D'Alembert (1717–1783), long the *secrétaire perpétuel* of the French Academy and with Denis Diderot the leading spirit of the famous *Encyclopédie ou dictionnaire raisonné des sciences, des arts et des métiers* (28 vols.; Paris, 1751–1772). In this *Encyclopédie* D'Alembert wrote a number of articles,[1] and in the article entitled "Différentiel" (vol. 4, 1754) he came to the expression of the derivative as the limit of a quotient of increments, that is, of what we now write $dy/dx = \lim \Delta y/\Delta x, \Delta x \to 0$ (already, though not in a very clear way, expressed by Newton). This leading idea, however, was not followed up immediately, either by D'Alembert himself or by others. One of the difficulties that prevented acceptance of the limit concept in this case was of the same nature as the Zeno paradoxes: how can a limit be reached if the process of coming to it consists of an infinite number of steps? Only with Cauchy in the early

[1] For an account of several of them see G. Loria in the *Actes . . . du 3e congrès international d'histoire des sciences, tenu au Portugal en 1934* (Lisbon, 1935), 15 pp.