the 19th century. Today he is probably best known not for his work on the calculus of variations, the theory of numbers, or algebra, but for his search for sound foundations for the calculus. In his *Theory of Analytical Functions* (1797), which contained his lectures given at the École Polytechnique, he attempted to demonstrate that Taylor's power series expansions alone were sufficient to provide the sought after satisfactory foundations for the calculus. In a supplementary article entitled "Lessons on the Calculus of Functions" (1801), he introduced a new symbolism for first derivative f', for second derivative f'', and so on. In the 19th century, Ampère, Cauchy, Weierstrass, and others successfully extended his search for sound foundations.

86. From "Attempt at a New Method for Determining the Maxima and Minima of Indefinite Integral Formulas" (1760–61)*

(The Calculus of Variations)

JOSEPH-LOUIS LAGRANGE

The first problem of this kind solved by the geometers is that of the *Brachystochrone*, or line of most rapid descent, which Mr. Jean Bernoulli proposed toward the end of the last century. It was solved only for particular cases, and it was not until some time later, on the occasion of the investigations on *Isoperimetrics*, that the great geometer whom we mentioned and his illustrious brother Mr. Jacques Bernoulli gave some general rules for solving several other problems of the same kind. But since these rules were not general enough, all these investigations were reduced by the famous Mr. Euler to a general method, in a work entitled *Methodus inveniendi . . .*, an original work which everywhere radiates a deep knowledge of the calculus. But, however ingenious and fertile his method may be, we must recognize that it does not have all the simplicity that might be desired in a subject of pure analysis. The author has made us aware of this in Article 39 of Chapter II of his book, by the words, "A method free from a geometric solution is therefore required . . ."

Now here is a method that demands only a very simple application of the principles of the differential and integral calculus, but first of all I must warn you that, since this method demands that the same quantities vary in two different manners, I have, in order not to confuse these variations, introduced into my calculations a new characteristic δ. Thus δZ will express a difference of Z that will not be the same as dZ, but that nevertheless, will be formed by means of the same rules; so

**Source:* The French original of this paper, "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies," is in the *Oeuvres des Lagrange*, I (1867), 355-362. This English translation of parts of it is taken from D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (1969), 407-410 and 412-413. It is reprinted by permission of Harvard University Press.

that when we have an equation $dZ = m\,dx$ we might just as well have $\delta Z = m\delta x$, and other expressions in the same way.

This being settled, I come first to the following problem.

I

Problem I. Given an indefinite integral expression represented by $\int Z$, where Z indicates a given arbitrary function of the variables x, y, z and their differentials [*différences*] $dx, dy, dz, d^2x, d^2y, d^2z, \ldots$, to find the relation among these variables so that the formula $\int Z$ become a maximum or a minimum.

Solution. According to the known method *de maximis et minimis* we shall have to differentiate the proposed $\int Z$, and, regarding the quantities x y, z, dx, dy, dz, $d^2x, d^2y, d^2z, \ldots$ as variables, make the resulting differential [*différentielle*] equal to zero. When, therefore, we indicate these variations by δ, we shall have first, for the equation of the maximum or minimum,

$$\delta \int dZ = 0,$$

or, what is equivalent to it,

$$d \int \delta Z = 0.$$

Now, let Z be such that

$$\begin{aligned}\delta Z = {} & n\delta x + p\delta\, dx + q\delta\, d^2x + r\delta\, d^3x + \cdots \\ & + N\delta y + P\delta\, dy + Q\delta\, d^2y + R\delta\, d^3y + \cdots \\ & + \nu\delta z + \pi\delta\, dz + \chi\delta\, d^2z + \rho\delta\, d^3z + \cdots;\end{aligned}$$

then we obtain from it the equation

$$\begin{aligned}& \int n\delta x + \int p\delta\, dx + \int q\delta\, d^2x + \int r\delta\, d^3x + \cdots \\ & + \int N\delta y + \int P\delta\, dy + \int Q\delta\, d^2y + \int R\delta\, d^3y + \cdots \\ & + \int \nu\delta z + \int \pi\delta\, dx + \int \chi\delta\, d^2z + \int \rho\delta\, d^3z + \cdots = 0,\end{aligned}$$

but it is easily understood that

$$\delta\, dx = d\delta\, x, \quad \delta\, d^2x = d^2\delta x,$$

and the others in the same way; moreover, we find by the method of integration by parts,

$$\begin{aligned}& \int p\, d\delta x = p\delta x - \int dp\delta x, \\ & \int q\, d^2\delta x = q\, d\delta x - dq\delta x + \int d^2q\delta x, \\ & \int r\, d^3\delta x = r\, d^2\delta x - dr\, d\delta x + d^2r\delta x - \int d^3z\delta x,\end{aligned}$$

and the others in a similar way. The preceding equation will therefore be changed into the following:

$$\text{(A)}\quad \begin{aligned}& \int (n - dp + d^2q - d^3r + \cdots)\delta x \\ & + \int (N - dP + d^2Q - d^3R + \cdots)\delta y \\ & + \int (\nu - d\pi + d^2\chi - d^3\rho + \cdots)\delta z \\ & + (p - dq + d^2r - \cdots)\delta x + (q - dr + \cdots)d\delta x \\ & \qquad + (r - \cdots)d^2\delta x + \cdots \\ & + (P - dQ + d^2R - \cdots)\delta y + (Q - dR + \cdots)d\delta y \\ & \qquad + (R - \cdots)d^2\delta y + \cdots \\ & + (\pi - d\chi + d^2\rho - \cdots)\delta z + (\chi - d\rho + \cdots)\, d\delta z \\ & \qquad + (\rho - \cdots)d^2\delta z + \cdots = 0,\end{aligned}$$

from which we obtain first the indefinite equation

$$\text{(B)}\qquad \begin{aligned}&(n - dp + d^2q - d^3r + \cdots)\delta x\\ &+ (N - dP + d^2Q - d^3R + \cdots)\delta y -\\ &+ (\nu - d\pi + d^2\chi - d^3\rho + \cdots)\delta z = 0,\end{aligned}$$

and then the determinate equation

$$\text{(C)}\qquad \begin{aligned}&(p - dp + d^2r - \cdots)\delta x + (q - dr + \cdots)d\delta x + (r - \cdots)d^2\delta x + \cdots\\ &+ (P - dQ + d^2R - \cdots)\delta y + (Q - dR + \cdots)d\delta y + (R - \cdots)d^2\delta y + \cdots\\ &+ (\pi - d\chi + d^2\rho - \cdots)\delta z + (\chi - d\rho + \cdots)d\delta z + (\rho - \cdots)d^2\delta z + \cdots = 0.\end{aligned}$$

This equation refers to the last part of the integral $\int Z$; but we must observe that, since each of its terms, such as $p\delta x$, depends on an integration by parts of the formula $\int p\, d\delta x$, we may add to or subtract from it a constant quantity. The condition by which this constant must be determined is that $p\delta x$ must vanish at the point where the integral $\int p\, d\delta x$ begins; we must therefore take away from $p\delta x$ its value at this point. From this we obtain the following rule. Let us express the first part of equation (C) generally by M, and let the value of M at the point where the integral $\int Z$ begins be indicated by $'M$, and at the point where this integral ends, by M'; then we have $M' - 'M = 0$ for the complete expression of equation (C). Now, in order to free the equations obtained from the undetermined differentials $\delta x, \delta y, \delta z, d\delta x, d\delta y, \ldots$, we must first examine whether, by the nature of the problem, there exists some given relation among them, and then, having reduced them to the smallest number possible, we must equate to zero the coefficient of each of those that remain. If they are absolutely independent of each other, then equation (B) will give us immediately the three following:

$$\begin{aligned}n - dp + d^2q - d^3r + \cdots &= 0,\\ N - dP + d^2Q - d^3R + \cdots &= 0,\\ \nu - d\pi + d^2\chi - d^3\rho + \cdots &= 0.\end{aligned}$$

[Next follows the example

$$\int \frac{\sqrt{dx^2 + dy^2 + dx^2}}{\sqrt{x}},$$

which is the brachystochrone in empty space and leads *(a)* to the result that the curve is plane, and *(b)* to $dt = \sqrt{x}\, dx/\sqrt{c - x}$. The case of the brachystochrone on a surface is also discussed; here the relation $\delta z = p\delta x + q\delta y$ has to be taken into consideration. Lagrange takes the cases in which the end points are fixed, as well as those in which they are subjected to certain other conditions. This, says Lagrange, makes his method more general than that of Euler, since Euler keeps the end points fixed; moreover, he lets only y vary in Z. . . .]

Problem III. To find the equation of the maximum or the minimum of the formula $\int Z$, if Z is simply given by a differential equation that does not contain other differentials of Z than the first.

[This is the case in which we can write

$$\delta\, dZ + T\delta Z = n\delta x + p\delta\, dx + \cdots + N\delta y + P\delta\, dy + \cdots + \nu\delta z + \pi\delta\, dz,$$

which is then solved as a linear differential equation in δZ, taking $\delta\, dZ = d\delta Z$.

There are two appendices. In the first we find (a) the problem of the surface of least area among all surfaces with the same given perimeter:

$$\delta \int\int dx\, dy\sqrt{1 + p^2 + q^2} = 0, \qquad p = \left(\frac{dz}{dx}\right), \qquad q = \left(\frac{dz}{dy}\right),$$

which leads to the condition that both $p\, dx + q\, dy$ and $\dfrac{p\, dy - q\, dx}{\sqrt{1 + p^2 + q^2}}$ have to be exact differentials,[1] and *(b)* the problem of the surface of least area among all surfaces of equal volume:

$$\delta\left(\int\int z\, dx\, dy\right) = 0, \quad \delta\left(\int\int dx\, dy\sqrt{1 + p^2 + q^2}\right) = 0,$$

which leads to the condition that both $p\, dx + q\, dy$ and $\dfrac{p\, dy - q\, dx}{\sqrt{1 + p^2 + q^2}} + kx\, dy$ (k an arbitrary coefficient) must be exact differentials. This is verified for the sphere.

In the second appendix we find the problem of the polygon of largest area among all polygons of the same given number of sides. It is shown that this polygon is inscribed in a circle, a theorem proved geometrically by Cramer (*Histoire de l'Académie Royale, Berlin,* 1752). If only the sum of the sides is given, the polygon is regular.

Lagrange's paper was followed in the same number of the *Miscellanea Taurinensis,* pp. 196-298, by a longer one: "Application de différents problèmes de dynamique" (*Oeuvres,* I, 365-468).]

NOTE

1. Two examples of these "minimal surfaces," the catenoid and the right helicoid, were found by Jean-Baptiste Meusnier, a pupil of Monge's, in the *Mémoires des savants étrangers de l'Académie 10* (Paris, 1785). He also interpreted here Lagrange's analytic condition geometrically as indicating that the mean curvature is zero. The catenoid had already appeared in chap. V, 44 of Euler's *Methodus inveniendi,* but not as a minimal surface.