variable quantity a quantity composed in some way or other of this variable quantity and constants" (*Opera omnia*, II, 241). In this paper Bernoulli then used the term "function" quite freely in enunciating his theorems. Euler took it over and in a paper in *Commentarii Academiae Scientiarum Petropolitanae 7, 1734–35* (1740), 184–200 (*Opera omnia*, ser. I, vol. 22, 57–75) introduced the notation $f\left(\frac{x}{a} + c\right)$ for "an arbitrary function of $\frac{x}{a} + c$." In Chapter I of his *Introductio* of 1748 (see Selection V.15) Euler repeated Bernoulli's statement, adding the word "analytic," and continued, "Therefore every analytic expression in which apart from a variable quantity z all quantities that compose this expression are constants is a function of this z, such as $a + 3z$, $az - 4zz$, $az + baa - zz - c^2$, etc."

Euler then classified functions, using the terms "algebraic" and "transcendental," "single-valued" and "multiple-valued." In the second volume of the *Introductio* he discussed curved lines, and wrote (Chapter I):

"A continuous curve is of such a nature that it can be expressed by one definite function of x. But if a curved line is of such a nature that various parts of it, BM, MD, DN, etc., are expressed by various functions of x such that, after the part BM has been defined with the aid of one function, the part MD is described by another function, then we call such curved lines discontinuous or mixed and irregular, because they are not formed according to one constant law and are composed of parts of various continuous curves."

In his *Institutiones calculi differentialis* (Saint Petersburg, 1755), Euler returned to these statements in the *Introductio* and then showed how to differentiate these functions. It is clear, therefore, that in Euler's opinion (and in that of his contemporaries and pupils), a function was a relation to be expressed by some analytical expression, as a polynomial, a sine, a logarithm, or even an integral of such expressions.

It was the exchange of opinions among Euler and some of his colleagues due to the vibrating-string discussion that brought about a certain feeling of disturbance among those who used the concept of function in this way. As we have seen in Selection V.16, Taylor had shown that there are sinusoidal solutions. D'Alembert found the solution in the form $z = f(at + x) + f(at - x)$, with $f(x)$ an "arbitrary function," but was not sure that this "analytic way," as he called it, of expressing a solution was sufficient to describe all forms of the string in motion; in other words, he was not sure whether any continuous curve could be given by an expression $y = f(x)$. Euler thought that this could be done. But Daniel Bernoulli derived the solution in the form of an infinite trigonometric series and gave it as his opinion that this combination of "Taylorian" functions could give the general solution, something Euler doubted. Euler's conclusion was (Selection V.16(5)) that his trigometric solution was only a particular solution of the formula which in general contains all the curves that the string in motion can assume, and there are an infinity of other curves that cannot be expressed by this equation.

For Euler, "arbitrary functions" were able to represent all "curves of the string" and conversely. Later (1759) Lagrange argued that an arbitrary function in great generality can be expressed by a trigonometric series. On Lagrange's definition of function see Selection V.19.

As we have said, the concept of function was clarified in the nineteenth century by the work of Fourier, Cauchy, Dirichlet, and Riemann; see, for example, P. E. B. Jourdain, "The origins of Cauchy's conceptions of a definite integral and of the continuity of a function," *Isis 1* (1913), 661–703; A. Pringsheim in *Encyklopädie der mathematischen Wissenschaften* (Teubner, Leipzig), II (1899), 1–53.

17 LAMBERT. IRRATIONALITY OF π

By 1750 the number π had been expressed by infinite series, infinite products, and infinite continued fractions, its value had been computed by infinite series to 127 places of decimals (see Selection V.15), and it had been given its present symbol. All these efforts, however, had not contributed to the solution of the ancient problem of the quadrature of the circle; the question whether a circle whose area is equal to that of a given square can be constructed with the sole use of straightedge and compass remained unanswered. It was Euler's discovery of the relation between trigonometric and exponential functions that eventually led to an answer. The first step was made by J. H. Lambert, when, in 1766–1767, he used Euler's work to prove the irrationality not only of π, but also of e.

Johann Heinrich Lambert (1728–1777) was a Swiss from Mülhausen (then in Switzerland). Called to Berlin by Frederick the Great, he became a member of the Berlin Academy and thus a colleague of Euler and Lagrange. His name is also connected with the introduction of hyperbolic functions (1770), with perspective (1759, 1774), and with the so-called Lambert projection in cartography (1772).

Lambert published his proof of the irrationality of π in his "Vorläufige Kenntnisse für die, so die Quadratur und Rectification des Circuls suchen," *Beyträge zum Gebrauche der Mathematik und deren Anwendung 2* (Berlin, 1770), 140–169, written in 1766, and in more detail in the "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques," *Histoire de l'Académie, Berlin, 1761* (1768), 265–322, presented in 1767. They have been reprinted in the *Opera mathematica*, ed. A. Speiser (2 vols.; Füssli, Zurich, 1946, 1948), I, 194–212, II, 112–159. The following text is a translation from pp. 132–138 of vol. II. Lambert writes tang where we write tan. See also F. Rudio, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung* (Teubner, Leipzig, 1892).

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37. *Now I say that this tangent* [tan φ/ω] *will never be commensurable to the radius, whatever the integers* ω, φ *may be.*[1]

[1] In the previous sections Lambert expands tan v, v an arbitrary arc of a circle of radius 1, into a continued fraction, and gets for $v = 1/w$

$$\tan v = \cfrac{1}{w - \cfrac{1}{3w - \cfrac{1}{5w - 1}}} \quad \text{etc.}$$

Investigating the partial fractions and their residues, he finds infinite series like

$$\tan v = \frac{1}{w} + \frac{1}{w(3w^2 - 1)} + \frac{1}{(3w^2 - 1)(15w^3 - 6w)} + \cdots$$

and shows (in §34) that these series converge more rapidly than any decreasing geometric series. Then, if $w = \omega:\varphi$, ω, φ being relatively prime integers, he finds for the partial fractions of tan v (§36):

$$\frac{\varphi}{\omega}, \quad \frac{3\omega\varphi}{3\omega^2 - \varphi^2}, \quad \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\varphi^2\omega}, \quad \frac{105\omega^3\varphi - 10\omega\varphi^3}{105\omega^4 - 45\omega^2\varphi^2 + \varphi^2}, \quad \text{etc.,}$$

and (§37):

$$\tan\frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega^2\varphi)} + \text{etc.}$$

Then follows the text which we reproduce.

38. To prove this theorem, let us write

$$\tan\frac{\varphi}{\omega} = \frac{M}{P},$$

such that M and P are quantities expressed in an arbitrary way, even, if you like, by decimal sequences, which always can happen, even when M, P are integers, because we have only to multiply each of them by an irrational quantity. We can also, if we like, write

$$M = \sin\frac{\varphi}{\omega}, \quad P = \cos\frac{\varphi}{\omega},$$

as above. And it is clear that, even if $\tan \varphi/\omega$ were rational, this would not necessarily hold for $\sin \varphi/\omega$ and $\cos \varphi/\omega$.

39. Since the fraction M/P exactly expresses the tangent of φ/ω, it must give all the quotients w, $3w$, $5w$, etc., which in the present case are

$$+\frac{\omega}{\varphi}, \quad -\frac{3\omega}{\varphi}, \quad +\frac{5\omega}{\varphi}, \quad -\frac{7\omega}{\varphi}, \quad \text{etc.}$$

40. Hence, if the tangent of φ/ω is rational, then clearly M will be to P as an integer μ is to an integer π, such that, if μ, π are relatively prime, we shall have

$$M:\mu = P:\pi = D,$$

and D will be the greatest common divisor of M, P. And since reciprocally

$$M:D = \mu, \quad P:D = \pi,$$

we see that, since M, P are supposed to be irrational quantities, their greatest common divisor will be equally an irrational quantity, which is the smaller, the larger the quotients μ, π are.

41. Here are therefore *the two suppositions of which we must show the incompatibility*. Let us first divide P by M, and the quotient must be $\omega:\varphi$. But since $\omega:\varphi$ is a fraction, let us divide φP by M, and the quotient ω will be the φ-tuple of $\omega:\varphi$. It is clear that we could divide it by φ if we wished to do so. This is not necessary, since it will be sufficient that ω be an integer. Having thus obtained ω by dividing φP by M, let the residue be R'. This residue will equally be the φ-tuple of what it would have been, and that we have to keep in mind. Now, since $P:D = \pi$, an integer, we still have $\varphi P:D = \varphi\pi$, an integer. Finally, $R':D$ will also be an integer. Indeed, since

$$\varphi P = \omega M + R',$$

we shall have

$$\frac{\varphi P}{D} = \frac{\omega M}{D} + \frac{R'}{D}.$$

But $\varphi P:D = \varphi\pi$, $\omega M:D = \omega\mu$; hence

$$\varphi\pi = \omega\mu + \frac{R'}{D},$$

which gives

$$\frac{R'}{D} = \varphi\pi - \omega\mu = \text{integer},$$

which we shall call r', so that $R'/D = r'$. The residue of the first division will therefore still have the divisor D, the greatest common divisor of M, P.

42. Now let us pass to the second division. The residue R' being the φ-tuple of what it would have been if we had divided P instead of φP, we must take this into account by the second division, where we divide φM, instead of M, by R' in order to obtain the second quotient, which $= 3\omega:\varphi$. However, in order to avoid the fractional quotient here also, let us divide $\varphi^2 M$ by R', in order to have the quotient 3ω, an integer. Let the residue be R'', and we shall have

$$\varphi^2 M = 3\omega R' + R'';$$

hence, dividing by D,

$$\frac{\varphi^2 M}{D} = \frac{3\omega R'}{D} + \frac{R''}{D}.$$

But

$$\frac{\varphi^2 M}{D} = \varphi^2 m = \text{integer},$$

$$\frac{3\omega R'}{D} = 3\omega r' = \text{integer};$$

hence

$$\varphi^2 m = 3\omega r' + \frac{R''}{D},$$

which gives $R''/D = \varphi^2 m - 3\omega r'$ = an integer number, which we shall write $= r''$, so that

$$\frac{R''}{D} = r''.$$

Hence the greatest common divisor of M, P, R' is still of the second residue R''.

43. Let the next residues be R''', $R^{\text{iv}}, \ldots, R^n, R^{n+1}, R^{n+2}, \ldots$ which correspond to the φ-tuple quotients $5\omega, 7\omega, \ldots, (2n-1)\omega, (2n+1)\omega, (2n+3)\omega, \ldots$, and we have to prove in general that if two arbitrary residues R^n, R^{n+1}, in

immediate succession, still have D as divisor, the next residue R^{n+2} will have it too, so that, if we write

$$R^n : D = r^n,$$
$$R^{n+1} : D = r^{n+1},$$

where r^n and r^{n+1} are integers, we shall also have

$$R^{n+2} : D = r^{n+2},$$

an integer. This is the demonstration.

We omit this proof in §44, since the reasoning follows that of §42.

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45. Now we have seen that r', r'' are integers (§§41, 42), hence also r''', $r^{\text{iv}}, \dots, r^n, \dots$ to infinity will be integers. Hence any one of the residues R', R'', $R''', \dots, R^n, \dots$ to infinity will have D as common divisor. Let us now find the value of these residues expressed in M, P.

46. Every division provides us with an equation for this purpose, since we have

$$R' = \varphi P - \omega M,$$
$$R'' = \varphi^2 M - 3\omega R',$$
$$R''' = \varphi^2 R' - 5\omega R'', \quad \text{etc.}$$

But let us observe that in the existing case the quotients ω, 3ω, 5ω, etc. are alternately positive and negative and that the signs of the residues succeed each other in the order $- - + +$. These equations can therefore be changed into

$$R' = \omega M - \varphi P,$$
$$R'' = 3\omega R' - \varphi^2 M,$$
$$R''' = 5\omega R'' - \varphi^2 R',$$
$$\dots\dots\dots$$

or in general

$$R^{n+2} = (2n - 1)R^{n+1} - \varphi^2 R^n.$$

From this we see that every residue is related to the two preceding in the same way as the numerators and denominators of the fractions that approximate the value of $\tan \varphi/\omega$ (§36).

47. Let us make the substitutions indicated by these equations in order to express all these residues by M, P. We shall have

$$R' = \omega M - \varphi P,$$
$$R'' = (3\omega^2 - \varphi^2)M - 3\omega\varphi P,$$
$$R''' = (15\omega^3 - 6\omega\varphi^2)M - (15\omega^2\varphi - \varphi^3)P, \quad \text{etc.}$$

And since these coefficients of M, P are the denominators and numerators of the fractions we found above for $\tan \varphi/\omega$ (§36), we see also that we shall have

$$\frac{M}{P} - \frac{\varphi}{\omega} = \frac{R'}{\omega P},$$
$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{R''}{(3\omega^2 - \varphi^2)P},$$
$$\frac{M}{P} - \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\omega\varphi^2} = \frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P}, \quad \text{etc.}$$

48. But we have

$$\frac{M}{P} = \tan\frac{\varphi}{\omega};$$

hence (§§37, 34)

$$\frac{M}{P} - \frac{\varphi}{\omega} = \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^2 - 6\omega\varphi^2)} + \text{etc.},$$
$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} + \text{etc.};$$

hence

$$\frac{R'}{\omega P} = \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} + \text{etc.},$$
$$\frac{R''}{(3\omega^2 - \varphi^2)P} = \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} + \text{etc.},$$
$$\frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P} = \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2)(105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} + \text{etc.}$$

Thus all the residues can be found by means of the sequence of differences (§37)

$$\tan\frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)}$$
$$+ \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2)(105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} + \text{etc.}$$

by omitting 1, 2, 3, 4, etc. of the first terms and multiplying the sum of the following terms by the first factor of the denominator of the first term that is retained and by P.

49. Now, this sequence of differences is more convergent than a decreasing geometric progression (§§34, 35). Hence the residues R', R'', R''', etc. decrease in such a way that they become smaller than any assignable quantity. And as every one of these residues, having D as common divisor, is a multiple of D, it follows that this common divisor D is smaller than any assignable quantity, which makes $D = 0$. Consequently $M:P$ is a quantity incommensurable with unity, hence irrational.

50. Hence *every time that a circular arc* $= \varphi/\omega$ *is commensurable with the radius* $= 1$, *hence rational, the tangent of this arc will be a quantity incommensurable with the radius, hence irrational. And conversely, every rational tangent is the tangent of an irrational arc.*

51. Now, since the tangent of 45° is rational, and equal to the radius, the arc of 45°, and hence also the arc of 90°, 180°, 360°, is incommensurable with the radius. Hence *the circumference of the circle does not stand to the diameter as an integer to an integer.* Thus we have here this theorem in the form of a corollary to another theorem that is infinitely more universal.

52. Indeed, it is precisely this absolute universality that may well surprise us.

Lambert then goes on to draw consequences from his theorem concerning arcs with rational values of the tangent. Then he draws an analogy between hyperbolic and trigonometric functions and proves from the continued fraction for $e^u + 1$ that e and all its powers with integral exponents are irrational, and that all rational numbers have irrational natural logarithms. He ends with the sweeping conjecture that "no circular or logarithmic transcendental quantity into which no other transcendental quantity enters can be expressed by any irrational radical quantity," where by "radical quantity" he means one that is expressible by such numbers as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt{2 + \sqrt{3}}$, and so forth. Lambert does not prove this; if he had, he would have solved the problem of the quadrature of the circle. The proof of Lambert's conjecture had to wait for the work of C. Hermite (1873), and F. Lindemann (1882). See, for instance, H. Weber and J. Wellstein, *Encyklopädie der Elementar-Mathematik* (3rd ed.; Teubner, Leipzig, 1909), I, 478–492; G. Hessenberg, *Transzendenz von e und π* (Teubner, Leipzig, Berlin, 1912); U. G. Mitchell and M. Strain, "The number e," *Osiris 1* (1936), 476–496.

18 FAGNANO AND EULER. ADDITION THEOREM OF ELLIPTIC INTEGRALS

Count Giulio Carlo de'Toschi di Fagnano (1682–1766), Spanish consul in his home town of Sinigaglia (Italy) and an amateur mathematician, published in the *Giornali de'letterati d'Italia* for the years 1714–1718 a series of papers on the summation of the arcs of certain

curves, a problem induced by a paper of Johann Bernoulli's of 1698.[1] These papers of Fagnano are reproduced in his *Opere mathematiche* (2 vols.; Albrighi, Segati & Co., Milan, Rome, Naples), II (1911), from which our selection has been translated. In vol. 19 of the *Giornali* Fagnano posed the following problem (*Opere*, II, 271):

Problem. Let a biquadratic primary parabola, which has as its constituent equation $x^4 = y$, and also a portion of it, be given. We ask that another portion of the same curve be assigned such that the difference of the two portions be rectifiable.

It had already been recognized by the brothers Bernoulli that what would be called elliptic arcs are not rectifiable, but that sums or differences might be representable by arcs of circles or straight lines. Fagnano gave a solution of his own problem, and generalized it to a number of cases, all involving elliptic integrals. One of his conclusions, sometimes called Fagnano's theorem, dates from 1716 and is found in the paper entitled "Teorema da cui si deduce una nuova misura degli archi elittici, iperbolici, e cicloidali," *Giornali 26* (*Opere*, II, 287–292).

Theorem. In the two polynomials below, X and Z, and in equation (1) the letters h, l, f, g represent arbitrary constant quantities.

I say, in the first place, that if in equation (1) the exponent s expresses the positive unity [$s = +1$], then the integral of the polynomial $X - Z$ is equal to $-hxz/\sqrt{-fl}$.

I say, in the second place, that if in the same equation (1) the exponent s expresses the negative unity [$s = -1$], then the integral of

$$X + Z = \frac{xz\sqrt{-h}}{\sqrt{g}}.$$

Here

$$X = \frac{dx\sqrt{hx^2 + l}}{\sqrt{fx^2 + g}},$$

$$Z = \frac{dz\sqrt{hz^2 + l}}{\sqrt{fz^2 + g}},$$

(1) $$(fhx^2z^2)^s + (flx^2)^s + (flz^2)^s + (gl)^s = 0.$$

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[1] An account of the contributions of Fagnano to this problem can be found in Cantor, *Geschichte*, III (2nd. ed., 1901), 465–472. Johann Bernoulli's paper, entitled "Theorema universale rectificationi linearum curvarum inserviens" (Universal theorem useful for the rectification of curved lines), appeared in the *Acta Eruditorum* of October 1698 (*Opera omnia*, I, 249–253); in it he asked whether there are curves with arcs that are not rectifiable, but are such that sums or differences of arcs are rectifiable. He claims that the parabola $3a^2y = x^3$ has that property. See Selection V.10, note 4.