14.D4 Joseph Louis Lagrange on solvability by radicals

[1] With regard to the solution of literal equations there has scarcely been any advance since the time of Cardano, who was the first to publish one for the equations of the third and fourth degrees. The first success of the Italian analysts in this subject seems to have been the end of the discoveries one can make there; at least it is certain that all the attempts which have been made to push back the limits of this part of algebra have still only served to find new methods for equations of the third and fourth degrees which do not appear to be applicable in general to equations of higher degree.

I propose in this memoir to examine the different methods which have been found up till now for the algebraic solution of equations, to reduce them to general principles, and to examine *a priori* why these methods succeed with the third and fourth degrees but fail for higher degrees.

This examination will have a two-fold advantage: on the one hand it will serve to shed greater light on the known solutions to the third and fourth degrees; on the other hand it will be useful to those who wish to occupy themselves with the solution of higher degrees, in providing them with different views of this object and above all in saving them a great number of steps and useless attempts. [...]

[2] We conclude our analysis of the methods which concern the solution of equations of the fourth degree here. Not only have we related these methods to one another and show their interconnections and their mutual dependence, but we have also, and this is the principal point, given the *a priori* reason why they lead, some to resolvents of the third degree, others to resolvents of the sixth, but which can be reduced to the third. One has seen how this derives in general from the fact that the roots of these resolvents are functions of quantities x', x'', x''', x^{iv}, which, on making all the possible permutations of these four quantities, only receive three different values, like the function $x'x'' + x'''x^{\text{iv}}$, or six values of which two are equal and of opposite sign, like the function $x' - x'' - x''' - x^{\text{iv}}$, or even six values which, on dividing them into three pairs and taking the sum or the product of the values of each pair, the three sums or the three products are always the same, whatever permutation one makes of the quantities x', x'', x''', x^{iv},[...]. It is precisely the existence of such functions on which the solution of equations of the fourth degree depends. [...]

[3] It follows from these reflections that it is very doubtful if the methods of which we have been speaking can give a complete solution of equations of the fifth degree, and still more so to those of higher degrees. And this uncertainty, coupled with the length of the calculations which these methods involve, must repel in advance all those who would seek to use them to solve one of the most famous and important problems in algebra. Also we observe that the authors of these methods have themselves been content to apply them to the third and fourth degrees and that no one has yet undertaken to push their work further.

It would therefore be very desirable if one could judge *a priori* the success that one can expect in applying these methods to degrees higher than the fourth. We are going to try and give the means for this by an analysis similar to that which has served us up till now in respect of the known methods for the solutions of equations of the third and fourth degree.

14.D5 Joseph Louis Lagrange's additions to Euler's *Algebra*

The geometricians of the last century paid great attention to the Indeterminate Analysis, or what is commonly called the *Diophantine Algebra*; but Bachet and Fermat alone can properly be said to have added any thing to what Diophantus himself has left us on that subject.

To the former we particularly owe a complete method of resolving, in integer numbers, all indeterminate problems of the first degree: the latter is the author of some methods for the resolution of indeterminate equations, which exceed the second degree; of the singular method, by which we demonstrate that it is impossible for the sum, or the difference of two biquadrates to be a square; of the solution of a great number of very difficult problems; and of several admirable theorems respecting integer numbers, which he left without demonstration, but of which the greater part has since been demonstrated by M. Euler in the Petersburg *Commentaries.*

In the present century, this branch of analysis has been almost entirely neglected; and, except M. Euler, I know no person who has applied to it: but the beautiful and numerous discoveries, which that great mathematician has made in it, sufficiently compensate for the indifference which mathematical authors appear to have hitherto entertained for such researches. The *Commentaries* of Petersburg are full of the labours of M. Euler on this subject, and the preceding Work is a new service, which he has rendered to the admirers of the *Diophantine Algebra*. Before the publication of it, there was no work in which this science was treated methodically, and which enumerated and explained the principal rules hitherto known for the solution of indeterminate problems. The preceding Treatise unites both these advantages: but, in order to make it still more complete, I have thought it necessary to make several Additions to it, of which I shall now give a short account.

The theory of Continued Fractions is one of the most useful in arithmetic, as it serves to resolve problems with facility, which, without its aid, would be almost unmanageable; but it is of still greater utility in the solution of indeterminate problems, when integer numbers only are sought. This consideration has induced me to explain the theory of them, at sufficient length to make it understood. As it is not to be found in the chief works on arithmetic and algebra, it must be little known to mathematicians; and I shall be happy, if I can contribute to render it more familiar to them. At the end of this theory, which occupies the first Chapter, follow several curious and entirely new problems, depending on the truth of the same theory; but which I have thought proper to treat in a distinct manner, in order that the solution of them may become more interesting. Among these will be particularly remarked a very simple and easy method of reducing the roots of equations of the second degree to Continued Fractions, and a rigid demonstration, that those fractions must necessarily be always periodical.

The other Additions chiefly relate to the resolution of indeterminate equations of the first and second degree; for these I give new and general methods, both for the case in which the numbers are only required to be rational, and for that in which the numbers sought are required to be integer; and I consider some other important matters relating to the same subject.

The last Chapter contains researches on the functions, which have this property, that the product of two or more similar functions is always a similar function. I give a general method for finding such functions, and shew their use in the resolution of different indeterminate problems, to which the usual methods could not be applied.

Such are the principal objects of these Additions, which might have been made much more extensive, had it not been for exceeding proper bounds; I hope, however, that the subjects here treated will merit the attention of mathematicians, and revive a taste for this branch of algebra, which appears to me very worthy of exercising their skill.

Chapter VIII

Remarks on Equations of the form $p^2 = Aq^2 + 1$, and on the common method of resolving them in Whole Numbers.

The method of Chap VII of the preceding Treatise, for resolving equations of this kind, is the same that Wallis gives in his *Algebra* (Chap. XCVIII), and ascribes to Lord Brouncker. We find it, also in the *Algebra* of Ozanam, who gives the honour of it to M. Fermat. Whoever was the inventor of this method, it is at least certain, that M. Fermat was the author of the problem which is the subject of it. He had proposed it as a challenge to all the English mathematicians, as we learn from the *Commercium Epistolicum* of Wallis; which led Lord Brouncker to the invention of the method in question. But it does not appear that this author was fully apprised of the importance of the problem which he resolved. We find nothing on the subject, even in the writings of Fermat, which we possess, nor in any of the works of the last century, which treat of the Indeterminate Analysis. It is natural to suppose that Fermat, who was particularly engaged in the theory of integer numbers, concerning which he has left us some very excellent theorems, had been led to the problem in question by his researches on the general resolution of equations of the form,

$$x^2 = Ay^2 + B,$$

to which all quadratic equations of two unknown quantities are reducible. However, we are indebted to Euler alone for the remark, that this problem is necessary for finding all the possible solutions of such equations.

The method which I have pursued for demonstrating this proposition is somewhat different from that of M. Euler; but it is, if I am not mistaken, more direct and more general. For, on the one hand, the method of M. Euler naturally leads to fractional expressions, where it is required to avoid them; and, on the other, it does not appear very evidently, that the suppositions, which are made in order to remove the fractions, are the only ones that could have taken place. Indeed, we have elsewhere shewn, that the finding of one solution of the equation $x^2 = Ay^2 + B$, is not always sufficient to enable us to deduce others from it, by means of the equation $p^2 = Aq^2 + 1$; and that, frequently, at least when B is not a prime number, there may be values of x and y, which cannot be contained in the general expressions of M. Euler.

With regard to the manner of resolving equations of the form $p^2 = Aq^2 + 1$, I think that of Chap. VII, however ingenious it may be, is still far from being perfect. For, in the first place, it does not shew that every equation of this kind is always resolvable in whose numbers, when A is a positive number not a square. Secondly, it is not demonstrated, that it must always lead to the solution sought for. Wallis, indeed, has professed to prove the former of these propositions; but his demonstration, if I may presume to say so, is a mere *petitio principii* (see Chap. XCIX). Mine, I believe, is the first rigid demonstration that has appeared. It is in the *Mélanges de Turin*, Vol. IV; but it is very long, and very indirect: that of Art. 37 is founded on the true principles of the subject, and leaves, I think, nothing to wish for. It enables us, also to appreciate that of Chap. VII, and to perceive the inconveniences into which it might lead, if followed without precaution. This is what we shall now discuss. [...]

I believe I had, at the same time with M. Euler, the idea of employing the irrational, and even imaginary factors of formulae of the second degree, in finding the conditions, which render those formulae equal to squares, or to any powers. On this subject, I read a Memoir to the Academy in 1768, which has not been printed; but of which I have given a summary at the end of my researches *On Indeterminate Problems*, which are to be found in the volume for the year 1767, printed in 1769, before even the German edition of M. Euler's *Algebra*.

In the place now quoted, I have shewn how the same method may be extended to formulae of higher dimensions than the second; and I have by these means given the solution of some equations, which it would perhaps have been extremely difficult to resolve in any other way. It is here intended to generalize this method still more, as it seems to deserve the attention of mathematicians, from its novelty and singularity.