13.B1 Bernoulli's lecture to l'Hôpital on the solution to Debeaune's problem

Another such example is the problem set to M. Descartes by M. Debeaune, the solution to which is not in his works but can be found in his *Letters* (Vol. III, No. 71). The solution of it does not appear to be very easy according to our method, indeed at first sight the problem appears impossible by this method [separation of variables]. But we shall see that by a change of variables it becomes easy to separate them, and that this problem can be solved completely once the quadrature of the hyperbola is given, for the curve is mechanical.

The problem goes like this: a line AC makes an angle of half a right angle with the axis AD, and E is a given constant line segment; what is the nature of the curve AB in which the ordinates BD are to the subtangents FD as the given E is to BC? *Solution.* Let $AD = x$, $DB = y$, $E = a$, suppose by hypothesis that $dy{:}dx = a{:}(y - x)$, then $adx = ydy - xdy$. From this equation the nature of the curve is to be found, either by integration or by rewriting y with dy on one side and x with dx on the other, for then two areas can be found and by comparing them the nature of the curve can be found. But the equation just found cannot be integrated, nor can x and dx be separated from y and dy; however, it can be changed into another by substituting the value of another variable. Therefore let $y - x = z$, $y = x + z$ and $dy = dz + dx$. The equation just found transforms into this: $adx = zdz + zdx$ or $adx - zdx = zdz$ and $dx = zdz{:}(a - z)$. Therefore these two variables separate, and we are led to the curve on multiplying by a, $adx = azdz{:}(a - z)$. And dropping normals GT and NH, $GN = GH = a$, and drawing through the points H and N HV and MR parallel to GT, $NR = NG$; erecting a perpendicular RS and asymptotes RM, RS and drawing a hyperbola LKG through G: then $GO = z$ & $GQ = x$, and $KO = az{:}(a - z)$, and because QI always equals a, the hyperbolic space KGO will equal the rectangle HQ & producing the lines IQ, KO, the point P where they meet draws out the curve GPW which satisfies the equation just found $adx = azdz{:}(a - z)$. Having constructed the curve AB there is then no more work; for, QP being produced to Z [Figure 1], as PZ shall equal the abscissa GQ the point Z will lie on the curve AB. Since $PZ = GQ = x = AD$ [Figure 2] and $QP = z$ [Figure 3], $QP + PZ$ will $= z + x = y = DB$ [Figure 2]. Q.E.I.

Corollary I NR [Figure 3] is asymptotic to GPW & $QP = BC$ [Figure 2]. This curve AB has its asymptote parallel to AC.

Corollary II The space $ADB = xy + ax - \frac{1}{2}yy$.

13.B2 Bernoulli on the integration of rational functions

Let the differential be $pdx{:}q$, of which p and q express rational quantities composed arbitrarily of a single variable x and constants; one seeks the integral or the algebraic sum or the means of reducing it to the quadrature of the hyperbola or the circle, the one or the other always being possible.

Let p be divided by q until the highest power of x in the remainder shall be less than q,

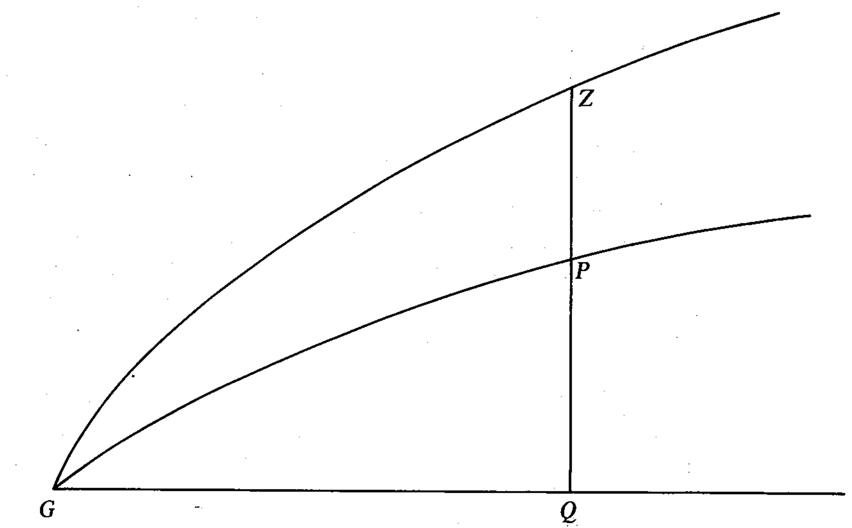

Figure 1

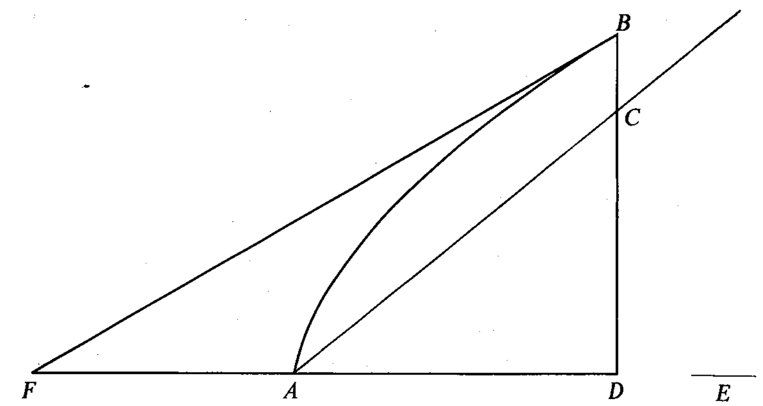

Figure 2

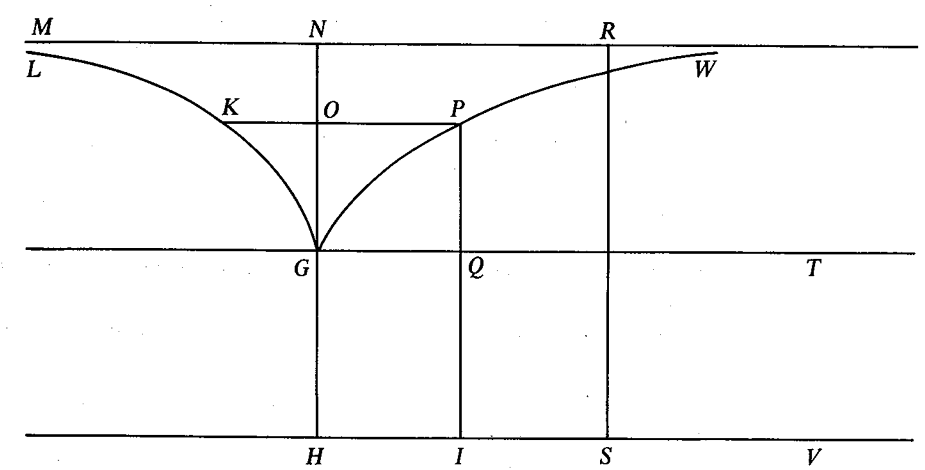

Figure 3

assuming the highest power of x in p is not already less than q, in which case there is no division to do. Then take the integral of the quotient of this division, which is always possible because this quotient (in respect of x) is always integral and rational.

Then for the integral of the remainder (which is properly the source of the difficulty) here is what one finds. Let the remainder be called r, and suppose that $rdx:q = adx:(x+f) + bdx:(x+g) + cdx:(x+h) +$ etc., that is to say, $rdx:q$ being equal to a collection of logarithmic differentials, that the highest power of x in q is unity. Here it is to be noted that a, b, c etc., and even f, g, h etc. are indeterminate constant quantities. [He goes on to explain how to find them by two methods. The second beings 'Let $rdx:q = sdx:t + adx:(x+f)$, taking for t a quantity involving x' and then argues by induction on the degree of t, which is one less than the degree of q. He then concludes as follows.]

One knows that $dx:(x+f)$, $dx:(x+g)$, $dx:(x+h)$ etc. are the differentials of logarithms of $x+f, x+g, x+h$ etc. And therefore that $\int(dx:(x+f)), \int(dx:(x+g)), \int(dx:(x+h))$ are logarithms themselves: in such a way that one will have $\int(dx:(x+f)) = l(x+f)$, and thus for the others, where l signifies logarithm as d signifies differential. Therefore $\int(adx:(x+f)) + \int(bdx:(x+g)) + \int(cdx:(x+h)) +$ etc. $= al(x+f) + bl(x+g) + cl(x+h) +$ etc. (by the nature of logarithms) $= \mathrm{Log}((x+f)^a \cdot (x+g)^b \cdot (x+h)^c$ etc.) Which was to be shown.

The abbreviated way of transforming compound differentials into simple ones and reciprocally; and even the imaginary simples into real compounds.

Problem I Transform the differential $adz:(bb-zz)$ into a logarithmic differential $adt:2bt$ and reciprocally.

Set $z = (t-1)b:(t+1)$, and you will have $adz:(bb-zz) = adt:2bt$. Reciprocally take $t = (+z+b):(-z+b)$ and you will have $adt:2bt = adz:(bb-zz)$.

Corollary One transforms the differential $adz:(bb+zz)$ in the same way into $-adt:2bt\sqrt{-1}$, an imaginary logarithmic differential, and reciprocally.

Problem II Transform the differential $adz:(bb+zz)$ into the differential of a sector or circular arc $-adt:2\sqrt{(t-bbtt)}$; and reciprocally.

Set $z = \sqrt{(1:t-bb)}$ and you will have $adz:(bb+zz) = -adt:2\sqrt{(t-bbtt)}$. Reciprocally take $t = 1:zz+bb$ and you will have $-adt:2\sqrt{(t-bbtt)} = adz:(bb+zz)$.

Problem III Transform the differential $adz:(bb-zz)$ into the differential of a hyperbolic sector $adt:2\sqrt{(t+bbtt)}$ and reciprocally.

Set $z = \sqrt{(1:t+bb)}$ and then $t = 1:(bb-zz)$; and you will have what is wanted.

Problem IV Transform the logarithmic differential $adt:2bt$ into the differential of a hyperbolic sector $adr:2\sqrt{(r+bbrr)}$.

Set $t = b + \sqrt{(1:r+bb)}:b - \sqrt{(1:r+bb)}$ and you will have what is wanted.

Corollary 1 One will transform the imaginary logarithmic differential $adt:2bt\sqrt{-1}$ into the differential of a real circular sector in the same way. For on setting $t = b\sqrt{-1} + \sqrt{(1:r-bb)}:b\sqrt{-1} - \sqrt{(1:r-bb)}$ one will have $adr:2\sqrt{(r-bbrr)}$.

Corollary 2 Then (Problem II) $\int(adz:(bb+zz))$ depends on the quadrature of the circle, and moreover $adz:(bb+zz)$ is $= \frac{1}{2}adz:(bb+bz\sqrt{-1}) + \frac{1}{2}adz:(bb-bz\sqrt{-1})$ which are two differentials of imaginary logarithms; one sees that imaginary logarithms can be taken for real circular sectors because the compensation which imaginary quantities make on being added together of destroying themselves in such a way that their sum is always real.

13.B3 Bernoulli on the inverse problem of central forces

The quadratures being supposed, and the centripetal law of forces being given arbitrarily in x and constants, to find the trajectory ABC which it makes a moving body describe.

Let $OA = a$, and from this radius the arc of the circle $AL = z$, $Ll = dz$, and in consequence $Nb = xdz:a$. Let also the time for Bb, proportional to $Nb \times BO$ (the double of triangle BOb), $= xxdz:a$. You know that this time, multiplied by the speed, i.e. [following the corollary to the preceding lemma] by $(ab - \int\phi dx)$ gives the space Bb. Therefore

$$xxdz\sqrt{(ab - \int\phi dx)}:a = Bb = \sqrt{(dx^2 + xxdz^2:aa)};$$

from which follows the equation

$$dz = aacdx:\sqrt{(abx^4 - x^4\int\phi dx - aaccxx)}$$

which expresses the nature of the sought-for trajectory ABC, and in which equation c is an arbitrary constant put in to make everything homogeneous. *That is what was to be found.*

You will see, Monsieur, that I have reached at a stroke a differential equation of the first degree, in which there is no mixing up of indeterminates; and so the geometric construction can be easily deduced from it, the quadratures of the curved spaces being given, and even more conveniently than Mr Newton found it in his *Principia*.

Moreover, my equation displays whether the sought-for trajectory is algebraic or not, depending on what hypothesis is given for the force. For if the integral

$$aacdx:\sqrt{(abx^4 - x^4\int\phi dx - aaccxx)}$$

is reducible to the arc of a circle whose radius stands to $OA(a)$ as number to number, then the sought-for curve will necessarily be algebraic. Thus the usual hypothesis of centripetal forces acting as the square of the reciprocal of the distance of the moving body from the centre, i.e. the hypothesis $\phi = aag:xx$, changes the preceding equation to

$$dz = aacdx:\sqrt{(abx^4 + aagx^3 - aaccxx)}$$
$$= aacdx:x\sqrt{(abxx + aagx - aacc)}$$

which can be reduced to such an arc of a circle; I see at once that your curve ABC must be algebraic on this hypothesis.