Hamilton's colleagues knew him as a genial man of exceptional intellect and broad scholarly interests. His candid disposition, good sense of humor, and eloquence appealed to many. His lectures attracted large audiences because of their literary merit, but his poetry was so poor that Wordsworth, who was a friend, urged him to confine himself to writing mathematics! Hamilton was fond of reading Plato and Kant. Coleridge had introduced him to Kant's writings, which in turn led him toward philosophical idealism. In the theory of matter he accepted the point–atomism of the Yugoslav scientist Rudjer Bŏsković.

Hamilton devoted much time and energy to building scientific organizations. After joining the Royal Irish Academy in 1835, he served as its president from 1837 to 1845. The chief organizer in Dublin of the British Association for the Advancement of Science, he brought its annual meeting there in 1835. He was knighted at that meeting, and the Royal Society twice awarded him the Royal (Gold) Medal for his optical and dynamical research. The St. Petersburg Academy of Science named him a corresponding member, and the newly-founded National Academy of Sciences in the United States placed him at the head of 14 foreign associates in 1863.

In his personal, domestic life Hamilton was not fortunate. Catherine Disney, whom he always loved, refused his proposal of marriage in 1825; so did Ellen DeVere in 1831. In 1833, he married Helen Bayly, who suffered from continued ill health, and they had two sons and a daughter. The sickly Helen ran a poor household; Hamilton rarely had regular meals. In 1853, when Catherine Disney neared death, he grew despondent. In his later years he struggled against alcoholism.

Hamilton contributed significantly to optics, dynamics, and, most of all, the algebra of quaternions. He first developed fundamental ideas in geometrical optics by employing the characteristic function, which involves the action of a system moving from its initial to its final point in space. He relied particularly on Fermat's principle of least time. Seeking the greatest generality, he extended the approach in 1833 and 1834 to provide a general method of dynamics. His general method not only unified optics and dynamics but also reduced dynamics to problems in the calculus of variations. His research included a detailed study of the three-body problem in astronomy. His new method was not fully appreciated for many years largely because of the novel, abstract, and obscure nature of his writings. The Hamiltonian method came into its own only with the rise of quantum mechanics.

Hamilton's landmark discovery of quaternions came in 1843. They are ordered sets of four ordinary numbers satisfying special laws of addition, multiplication, and equality but freeing algebra from the commutative law of multiplication (*i.e.*, $a \times b = b \times a$). The discovery followed a decade of patient and systematic research on algebra. Inspired by Kant's *Critique of Pure Reason*, Hamilton considered geometry to be the "science of pure space" and algebra to be based on the intuition of mathematical time. In a pioneer attempt to develop an axiomatic basis for algebra comparable to that of geometry, he developed a rigorous theory for complex numbers (*i.e.*, numbers of the form $a + bi$ where i is $\sqrt{-1}$). This allowed him to handle the two dimensional plane but not three-dimensional space. After examining triplets, he suddenly realized, while walking by Brougham Bridge in Dublin in 1843, that he needed quadruplets. The discovery so excited Hamilton that he carved the fundamental formulas in the bridge's stonework:

$$i^2 = j^2 = k^2 = i\,j\,k = -1.$$

Here was a three dimensional analogue of complex numbers to represent vectors in space, and he spent the final 22 years of his life developing and applying them. His results appeared in *Lectures on Quaternions* (1853) and the two-volume, posthumously published *Elements of Quaternions* (1866).

95. From *Elements of Quaternions* (1866)*

(On Quaternions, a Generalization of Complex Numbers)

WILLIAM ROWAN HAMILTON

108. Already we may see grounds for the application of the *name* QUATERNION, to such a *Quotient of two Vectors* as has been spoken of in recent articles. In the first place, such a quotient cannot *generally* be what we have called a SCALAR: or in other words, it cannot generally be equal to any of the (so-called) *reals of algebra,* whether of the *positive* or of the *negative* kind. For let x denote any such (actual)[1] scalar, and let α denote any (actual) vector; then we have seen that the product $x\alpha$ denotes *another* (actual) *vector,* say β', which is either *similar* or *opposite* in direction to α, according as the scalar coefficient, or *factor,* x, is positive or negative; in *neither* case, then, can it represent any vector, such as β, which is *inclined* to α, at any actual *angle,* whether acute, or right, or obtuse: or in other words, the *equation* $\beta' = \beta$, or $x\alpha = \beta$, is impossible, under the conditions here supposed. But we have agreed to write, as in algebra, $(x\alpha)/\alpha = x$; we must therefore[2] . . . *abstain* from writing *also* $\beta/\alpha = x$, under the same conditions: x still denoting a *scalar.* Whatever *else* a *quotient of two inclined vectors* may be found to be, it is thus, at least, a NON-SCALAR.

109. Now, in forming the conception of the *scalar itself,* as the *quotient of two parallel*[3] *vectors* we took into account not only *relative length,* or *ratio* of the usual kind, but also *relative direction,* under the form of *similarity or opposition.* In passing from α to $x\alpha$, we *altered* generally the *length* of the line α, in the ratio of $\pm x$ to 1; and we *preserved or reversed* the *direction* of that line, according as the *scalar coefficient* x was *positive* or *negative,* and, in like manner, in proceeding to form, more definitely than we have yet done, the conception of the *non-scalar quotient,* $q = \beta:\alpha$ = OB:OA, of *two inclined vectors,* which for simplicity may be supposed to be *co-initial,* we have *still* to take account both of the *relative length* and of the *relative direction,* of the two lines compared. But while the *former* element of the *complex relation* here considered, between these two lines or vectors, is *still* represented by a simple RATIO (of the kind commonly considered in geometry), or by a *number*[4] expressing that ratio; the *latter element* of

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the same complex relation is *now* represented by an ANGLE, AOB: and not simply (as it was before) by an *algebraical sign* + or −.

110. Again, in estimating this *angle*, for the purpose of *distinguishing* one quotient of vectors from another, we must consider not only its *magnitude* (or *quantity*), but also its PLANE: since otherwise, in violation of the principle[5] . . . , we should have *OB′:OA* = *OB:OA*, if *OB* and *OB′* were *two distinct rays* or sides of a *cone* of revolution, with OA for its *axis;* in which case . . . they would necessarily be *unequal vectors.* For a similar reason, we must attend also to the *contrast* between two *opposite angles,* of equal magnitudes, and in one *common plane.* In short, for the purpose of knowing *fully* the *relative direction* of two co-initial lines OA, OB in *space,* we ought to know not only *how many degrees* . . . the *angle* AOB contains; but also . . . the *direction of the rotation* from OA to OB: including a knowledge of the *plane, in which* the rotation is performed; and of the *band* (as *right* or *left,* when *viewed* from a known *side* of the plane), *towards which* the rotation is *directed.*

111. Or, if we agree to *select* some one *fixed band* (suppose the *right* hand), and to call all *rotations positive* when they are directed towards *this* selected hand, but all rotations *negative* when they are directed towards the *other band,* then, for *any given angle* AOB, supposed for simplicity to be less than two right angles, and considered as representing a *rotation in a given plane* from OA to OB, we may speak of *one perpendicular* OC *to that plane* AOB as being the *positive axis* of that rotation; and of the *opposite perpendicular* OC′ to the same plane as being the *negative axis* thereof: the rotation around the positive axis being *itself* positive, and *vice-versa.* And then the *rotation* AOB may be considered to be entirely *known,* if we know, first, its *quantity,* or the *ratio* which it bears to a *right rotation;* and second, the *direction* of its *positive axis,* OC, but not without knowledge of these *two* things, or of some data equivalent to them. But whether we consider the *direction of an* AXIS, or the *aspect of a* PLANE, we find (as indeed is well known) that the *determination* of such a *direction,* or of such an *aspect,* depend on TWO *polar coordinates,* or other *angular elements.*

112. It appears, then, from the foregoing discussion, that *for the complete determination,* of what we have called the *geometrical* QUOTIENT *of two coinitial Vectors,* a *System of Four Elements,* admitting each separately of numerical expression, *is generally required.* Of these four elements, *one* serves to determine the *relative length* of the two lines compared; and the other *three* are in general necessary, in order to determine *fully* their *relative direction.* Again, of these three latter elements, *one* represents the mutual *inclination,* or *elongation,* of the two lines; or the *magnitude* (or quantity) of the *angle* between them; while the *two others* serve to determine the *direction* of the *axis,* perpendicular to their common *plane,* round which a *rotation* through that angle is to be performed, in a *sense* previously selected as the *positive* one (or towards a fixed and previously selected *band*), for the purpose of *passing* (in the simplest way, and therefore in the plane of the two lines) from the *direction* of the *divisor-line,* to the direction of the *dividend-line.* And *no more than four* numerical *elements* are necessary for our present purpose: because the *relative length* of two lines is not changed when their lengths are altered proportionally, nor is their *relative direction* changed, when the *angle* which they form is merely *turned* about, *in its own plane.* On account, then, of this *essential connexion* of that *complex relation* between two lines, which is *compounded* of a *relation of lengths,* and of a *relation of directions,* and to which we have given (by an *extension* from the theory of *scalars*), the name of a *geometrical quotient,* with a *System of* FOUR *numerical Elements,* we have already a *motive* for saying that *"The Quotient of two Vectors is generally a Quaternion".*[6]

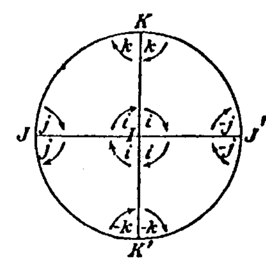
[*Text omitted.*]

181. Suppose that OI, OJ, OK are any three given and coinitial but rectangular unit lines, the rotation around the first from the second to the third being positive; and let OI′, OJ′, OK′ be the three unit vectors respectively opposite to these, so that

$$OI' = -OI,\ OJ' = -OJ,\ OK' = -OK.$$

Let the three new symbols i, j, k denote a *system of three right versors,*[7] *in three mutually rectangular planes,* . . . ; so that . . . i = OK:OJ, j = OI:OK, k = OJ:OI, as the figure may serve to illustrate. We shall then have these other expressions for the same three versors.

$$i = OJ':OK = OK':OJ' = OJ:OK';$$
$$j = OK':OI = OI':OK' = OK:OI';$$
$$k = OI':OJ = OJ':OI' = OI:OJ';$$



while the three respectively *opposite* versors may be thus expressed:

$$-i = OJ:OK = OK':OJ = OJ':OK' = OK:OJ'$$
$$-j = OK:OI = OI':OK = OK':OI' = OI:OK'$$
$$-k = OI:OJ = OJ':OI = OI':OJ' = OJ:OI'$$

and from the comparison of these different expressions several important symbolical consequences follow . . .

182. In the *first* place, since

$$i^2 = (OJ':OK).(OK:OJ) = OJ':OJ, \text{ etc.},$$

we deduce the following equal values for the *squares* of the new symbols:

I $\quad i^2 = -1;\ j^2 = -1;\ k^2 = -1,$

. . . .

In the *second* place, since

$$ij = (OJ:OK').(OK':OI) = OJ:OI, \text{ etc.},$$

we have the following values for the *products* of the same three symbols, or versors, when taken *two by two,* and in a certain *order* of *succession* . . . :

II $\quad ij = k;\ jk = i;\ ki = j.$

But in the *third* place . . . , since

$$ji = (OI:OK).(OK:OJ) = OI:OJ, \text{ etc.},$$

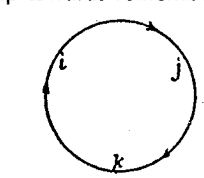
we have these other and *contrasted* formulae, for the *binary products* of the *same* three right versors, when taken as factors with an *opposite order:*

III $\quad ji = -k;\ kj = -i;\ ik = -j.$

Hence, while the *square of each* of the *three right versors,* denoted by these *three new symbols,* i, j, k, is equal to *negative unity,* the *product of any two* of them is equal either to the *third itself,* or to the *opposite* of that third versor, according as the *multiplier precedes* or *follows* the *multiplicand,* in the *cyclical succession*

i, j, k, i, j, . . .

which the annexed figure may give some help towards remembering.



183. Since we have thus ji = −ij, . . . we see that the *laws of combination of the new symbols,* i, j, k, are *not in all respects the same* as the corresponding laws in *algebra;* since the *Commutative Property of Multiplication,* or the *convertibility* of the places of the *factors* without change of value of the *product,* does *not here* hold good; which arises from the circumstance that the factors to be combined are here diplanar versors.

It is therefore important to observe that there is a respect in which the *laws of* i, j, k *agree* with usual and *algebraic laws:* namely, in the Associative Property of Multiplication; or in the property that the new symbols always obey the associative formula

$$\iota\kappa\lambda = \iota\kappa\lambda,$$

whichever of them may be substituted for ι, for κ, and for λ; in virtue of which equality of values we may *omit the point* in any such symbol of a *ternary product* (whether of equal or unequal factors), and write it simply as $\iota\kappa\lambda$. In particular, we have thus,

$$i.jk = i.i = i^2 = -1;$$
$$ij.k = k.k = k^2 = -1;$$

or briefly

$$ijk = -1.$$

We may, therefore, . . . establish the following important *Formula:*

$$i^2 = j^2 = k^2 = ijk = -1;$$

. . . which we shall find to contain (virtually) *all the laws of the symbols* i, j, k, and therefore to be a *sufficient symbolical basis* for the whole *Calculus of Quaternions:* because it will be shown that *every quaternion can be reduced to the Quadrinomial Form,*

$$q = w + ix + jy + kz,$$

where w, x, y, z compose a *system of four scalars,* while i, j, k are the same *three right versors as above.*

If two right versors in two mutually rectangular planes, be multiplied together in two opposite orders, the two resulting products will be two opposite right versors, in a third plane, rectangular to the two former; or in symbols . . .

$$q'q = -qq'$$

. . . In *this* case, therefore, we have what would be in algebra a *paradox* . . . When we come to examine what, in the last analysis, may be said to be the *meaning* of this last equation, we find it to be simply this: that *any two quadrantal* or *right rotations, in planes perpendicular to each other, compound themselves into a third right rotation, as their resultant in a plane perpendicular to each of them:* and that this *third* or *resultant* rotation has one or other of two *opposite directions,* according to the *order* in which the two *component rotations* are taken, so that one shall be *successive* to the other.

NOTES

1. Non zero.
2. By the second assumption.
3. Or collinear.
4. "The tensor of the quotient."
5. Assumption 2.
6. "Quaternion . . . signifies . . . a Set of Four."
7. A right versor is an operator which produces a rotation of a right angle about a given axis in a given direction.

Chapter VIII
The Nineteenth Century

Section A
Algebra

GEORGE BOOLE (1815–64)

George Boole, mathematician and logician, was born at Lincoln, England. His father, John Boole, was a cobbler deeply interested in elementary mathematics and the making of optical instruments. At an early age, the boy learned to assist his father and received his first lessons in mathematics from him. Young Boole attended a local elementary school and, briefly, a small commercial school. His favorite subject was classics. William Brooke, the owner of a scholarly circulating library, taught him Latin; Boole also learned Greek, French, and German on his own by age 14.

At age 15 Boole's future was seriously affected when his father's business declined. He set aside thoughts of taking holy orders and worked to help support his family. He began teaching in the village schools of the West Riding of Yorkshire. At age 20 he opened his own school in Lincoln. It was during these early years of teaching that Boole's talent for mathematics emerged. The Mechanics Institution (founded in Lincoln in 1834) had Royal Society publications in its reading room (Boole's father was curator of that room), and Boole devoted his scant leisure time to studying mathematics there. Almost unaided, he wrestled with Newton's *Principia*, Lagrange's *Mécanique analytique*, and Laplace's *Mécanique céleste*. He quickly earned a local reputation as a learned man. In 1835, at age 19, his first scientific publication, "An Address on the Genius and Discoveries of Sir Isaac Newton," was published.

Boole's work soon became known to a wider audience in the sciences. Beginning in 1839, he published a series of original papers on linear transformations and differential equations in the recently founded *Cambridge Mathematical Journal*. His papers from 1841 through 1843 on linear transformations generalized algebraic studies of Lagrange and Gauss on the relative invariance of discriminants and examined the absolute invariants of transformations. The limited results in these papers, representing a starting point of the theory of algebraic invariants, were extensively developed by one of their readers named Arthur Cayley. Unlike many of his contemporaries, Boole's treatment of differential equations went well beyond successful technical manipulation. He made the foundations of his subject more secure by avoiding the faulty use of analogy and insisting on precise definitions together with a rigorous process of reasoning. Using this approach he increased the power of the operational calculus. Boole also contributed to the *Philosophical Transactions* of the Royal Society, winning its Royal Medal in 1844 for