85. From *Introductio in analysin infinitorum* (1748)*

(Trigonometry)

LEONHARD EULER

ON TRANSCENDENTAL QUANTITIES WHICH CAN BE OBTAINED FROM THE CIRCLE

§126. After logarithms and exponential quantities we shall investigate circular arcs and their sines and cosines, not only because they constitute another type of transcendental quantities, but also because they can be obtained from these very logarithms and exponentials when imaginary quantities are involved.

Let us therefore take the radius of the circle, or its sinus totus, = 1. Then it is obvious that the circumference of this circle cannot be exactly expressed in rational numbers; but it has been found that the semicircumference is by approximation

$$= 3.14159.26535.89793\ldots$$

[127 decimal places are given[2]] for which number I would write for short

$$\pi,$$

so that π is the semicircumference of the circle of which the radius = 1, or π is the length of the arc of 180 degrees.[3]

§127. If we denote by z an arbitrary arc of this circle, of which I always assume the radius = 1, then we usually consider of this arc mainly the sine [*sinus*] and cosine [*cosinus*]. I shall denote the sine of the arc z in the future in this way

$$\sin. A.z, \quad \text{or only} \quad \sin. z$$

and the cosine accordingly

$$\cos. A.z, \quad \text{or only} \quad \cos. z.$$

Hence we shall have, since π is the arc of 180°,

$$\sin. 0 = 0, \quad \cos. 0 = 1$$

and

$$\sin. \tfrac{1}{2}\pi = 1, \quad \cos. \tfrac{1}{2}\pi = 0\ldots$$

[Now follows a whole set of trigonometric formulas including the definitions $\text{tang. } z = \dfrac{\sin. z}{\cos. z}$, $\cot. z = \dfrac{\cos. z}{\sin. z}$, the addition formulas, and identities such as

$$\text{tang. } \frac{a+b}{2} = \frac{\sin. a + \sin. b}{\cos. a + \cos. b}.$$

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Hereafter we omit the period after sin and cos and write i for $\sqrt{-1}$, as Euler also did in later work.[4]]

§132. Since

$$(\sin z)^2 + (\cos z)^2 = 1,$$

we shall have by factorization

$$(\cos z + i \sin z)(\cos z - i \sin z) = 1,$$

which factors, although imaginary [*etsi imaginarii*], still are of great use in combining and multiplying sines and cosines.

[Now comes De Moivre's theorem[5] (though the name is not mentioned), from which follows, in §133:]

$$\cos nz = \frac{(\cos z + i \sin z)^n + (\cos z - i \sin z)^n}{2}$$

and

$$\sin nz = \frac{(\cos z + i \sin z)^n - (\cos z - i \sin z)^n}{2i}.$$

When we develop these binomials in a series we shall get

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{1.2}(\cos z)^{n-2}(\sin z)^2 + \text{etc.}$$

and

$$\sin nz = \frac{n}{1}(\cos z)^{n-1}\sin z - \frac{n(n-1)(n-2)}{1.2.3}(\cos z)^{n-3}(\sin)^3 + \text{etc.}$$

§134. Let the arc z be infinitely small; then we get $\sin z = z$ and $\cos z = 1$; let now n be an infinitely large number, while the arc nz is of finite magnitude. Take $nz = v$; then since $\sin z = z = v/n$ we shall have

$$\cos v = 1 - \frac{v^2}{1.2.3} + \frac{v^4}{1.2.3.4} - \ldots + \text{etc.}$$

and

$$\sin v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \ldots + \text{etc.}$$

[Then, by writing $v = \dfrac{m}{n}\cdot\dfrac{\pi}{2}$ Euler obtains a series for $\sin \dfrac{m}{n}90°$ with terms up to $\dfrac{m^{29}}{n^{29}}$ and a series for $\cos \dfrac{m}{n}90°$ with terms up to $\dfrac{m^{30}}{n^{30}}$, the coefficients given to 28 decimals; these are followed by series for the tangent and the cotangent. He shows that it is only necessary to know the numerical values of these quantities for the values from 0° to 30° to be able to find them all by identities such as $\sin(30 + z) = \cos z - \sin(30 - z)$. Here cosec. z and sec. z are introduced.]

§138. Let us now take in the formulas of §133 the arc z infinitely small and let n be an infinitely small number ϵ [Euler writes i] such that ϵz will take the finite value v. We thus have $\epsilon z = v$ and $z = v/\epsilon$, hence $\sin z = v/\epsilon$ and $\cos z = 1$. After substituting these values we find

$$\cos v = \frac{\left(1 + \frac{vi}{\epsilon}\right)^{\epsilon} + \left(1 - \frac{vi}{\epsilon}\right)^{\epsilon}}{2},$$

$$\sin v = \frac{\left(1 + \frac{vi}{\epsilon}\right)^{\epsilon} - \left(1 - \frac{vi}{\epsilon}\right)^{\epsilon}}{2i}.$$

In the previous chapter we have seen that

$$\left(1 + \frac{z}{\epsilon}\right)^{\epsilon} = e^{z},$$

where by e we denote the base of the hyperbolic logarithms; if we therefore write for z first iv, then $-iv$, we shall have

$$\cos v = \frac{e^{iv} - e^{-iv}}{2}$$

and

$$\sin v = \frac{e^{iv} - e^{-iv}}{2i}.$$

From these formulas we can see how the imaginary exponential quantities can be reduced to the sine and cosine of real arcs. Indeed, we have

$$e^{iv} = \cos v + i \sin v,$$

$$e^{-iv} = \cos v - i \sin v.$$

[Then follow in §139 some formulas for the logarithms leading up to

$$z = \frac{1}{2i}\, l\, \frac{\cos z + i \sin z}{\cos z - i \sin z},$$

where l indicates logarithm.]

§140. Since $\frac{\sin z}{\cos z} = \text{tang } z$, the arc z can be expressed by its tangent in such a way that we have

$$z = \frac{1}{2i}\, l\, \frac{1 + i \text{ tang } z}{1 - i \text{ tang } z}.$$

Now we have seen above (§123) that

$$l\, \frac{1 + x}{1 - x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \text{etc.}$$

We now put $x = i$ tang z and shall obtain

$$z = \frac{\text{tang } z}{1} - \frac{(\text{tang } z)^3}{1} + \frac{(\text{tang } z)^5}{5} + \frac{(\text{tang } z)^7}{7} + \text{etc.}$$

If we therefore put tang $z = t$, so that z is the arc of which the tangent is t, which we shall indicate by A. tang. t [our $\tan^{-1} t$], we shall have

$$z = A. \text{ tang. } t.$$

Therefore, for known t, the corresponding arc will be

$$z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.}$$

Therefore, if the tangent t is equal to the radius 1, the arc $z = 45°$ or $z = \pi/4$, and we shall have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.},$$

which is the series first found by Leibniz to express the value of the circumference of the circle.

[The chapter ends with some other series for π that converge more rapidly.]

NOTES

1. [Does not appear in text reproduced here.]
2. Euler took this value from T. G. de Lagny, "Mémoire sur la quadrature du cercle," *Histoire de l'Académie Royale, Paris, 1719* (1727), 1e partie, 176–189, who computed π to 127 decimal places by means of a series for $\tan^{-1} 30°$.
3. The symbol π was never used in Antiquity; it seems first to have been used by William Jones (the editor of Newton's *Analysis per aequationes*, London, 1711) in his *Synopsis palmariorum matheseos* (London, 1706), p. 243. See D. E. Smith, *History of mathematics* (Ginn, New York, 1925), II, 312. Euler used π in his *Mechanica* (1736); see note 1 in his biography. See E. W. Hobson, *Squaring the circle* (Cambridge University Press, Cambridge, England, 1913). Euler, using the term *sinus totus* for the radius of the circle, adheres for the last time to the old terminology, in which the sine is a segment.
4. "In the following I shall denote the expression $\sqrt{-1}$ by the letter i so that $ii = -1$": Euler, *De formulis differentialibus angularibus*, presented to the Saint Petersburg Academy, 1777; published in the posthumous vol. IV of the *Institutiones calculi integralis* (1794), 183–194; *Opera omnia*, ser. I, vol. 19, 129–140, p. 130.
5. This theorem, now usually written $(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi$, appears at the opening of A. de Moivre, *Miscellanea analytica* (London, 1730), but in a different, more geometrical, form.

90. From "Demonstrations of Certain Arithmetical Theorems" (1738)*

(A proof of Fermat's great theorem—$x^n + y^n = z^n$ has no positive integral solutions for $n > 2$—for the case $n = 4$.)

LEONHARD EULER

Theorem 1. *The sum of two biquadratic numbers such as* $a^4 + b^4$ cannot be a square number unless one of the two biquadratic numbers vanishes.

Proof. I shall change the theorem to be demonstrated in such a way that I shall show that if in one case $a^4 + b^4$ were a square, no matter how large the numbers a and b, then I can progressively find smaller numbers a and b and at the end can reach the smallest integral numbers. Since there are no such smallest numbers of which the sum of the biquadratic numbers is a square, we must conclude that there are no such among the largest numbers.

Let therefore $a^4 + b^4$ be a square and a and b be relative primes, since if they were not relative primes, then I could reduce them by division to primes. Let a be an odd number; then b must be even, since necessarily one number must be even, the other one odd. Let us therefore write

$$a^2 = p^2 - q^2, \quad b^2 = 2pq;$$

here the numbers p and q must be relative primes, the one even, the other odd. But if $a^2 = p^2 - q^2$, then it is necessary that p be odd, because otherwise $p^2 - q^2$ could not be a square. Hence p is an odd number and q an even one. Since $2pq$ must also be a square it is necessary that both p and $2q$ be a square, because p and $2q$ are relative primes. Since $p^2 - q^2$ is a square, it is necessary that

$$p = m^2 + n^2 \quad \text{and} \quad q = 2mn,$$

where again m and n are relative prime numbers, of which one is even, the other odd. But since $2q$ is a square, $4mn$, or mn is a square, hence m and n are squares. If we therefore put

$$m = x^2, \quad n = y^2$$

then we shall have

$$p = m^2 + n^2 = x^4 + y^4,$$

which must equally be a square. From this it follows that if $a^4 + b^4$ were a square, then also $x^4 + y^4$ would be a square, but it is clear that the numbers x and y would be far smaller than a and b. In the same way we shall from the biquadratic numbers $x^4 + y^4$ again obtain smaller ones, of which the sum is a square, and we progressively reach the smallest biquadratic number among the integers. But since there are no smallest biquadratic numbers of which the sum gives a square, it is clear that there are no very large numbers either. However, if in one pair of the biquadratic numbers one of the terms is zero, then in all remaining pairs the one term vanishes, so that here nothing new results.

Corollary 1. Since therefore the sum of two biquadratic numbers cannot be a square, it is a fortiori impossible that the sum of two biquadratic numbers results in a biquadratic number.

Corollary 2. Although this demonstration pertains only to integers, yet it also shows that we cannot find among fractions two biquadratic numbers of which the sum is a square. Indeed, if $(a^4/m^4) + (b^4/n^4)$ were a square, then $a^4n^4 + b^4m^4$, which is a sum of integers, would also be a square, which we have proved to be impossible.

Corollary 3. By means of the same proof we can conclude that no numbers p and q exist such that p, $2q$ and $p^2 - q^2$ are squares; if such numbers existed then there would be values for a and b, which would render $a^4 + b^4$ square; for then $a = \sqrt{p^2 - q^2}$ and $b = \sqrt{2pq}$.

Corollary 4. Suppose therefore $p = x^2$ and $2q = 4y^2$, then $p^2 - q^2 = x^4 - 4y^4$. Then it could not at all happen, that $x^4 - 4y^4$ were a square. Nor could $4x^4 - y^4$ be a square; for then $16x^4 - 4y^4$ would be a square, which reduces it to the former case, because $16x^4$ is a biquadratic number.

Corollary 5. From this it follows that also $ab(a^2 + b^2)$ can never be a square. For the factors a, b, $a^2 + b^2$, all relative primes, would have to be squares, which is impossible.

Corollary 6. In a similar way, there cannot exist relatively prime numbers a and b such as to make $2ab(a^2 - b^2)$ a square. This follows from Corollary 3, where it was proven that no numbers p and q exist such as to make p, $2q$, $p^2 - q^2$ squares. And all this is valid also for numbers that are not relative primes, and the same for fractions according to Corollary 2.

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