upon trust, and believe points inconceivable? Whether they have not their mysteries, and what is more, their repugnancies and contradictions?

Qu. 65 Whether it might not become men who are puzzled and perplexed about their own principles, to judge warily, candidly and modestly concerning other matters?

Qu. 66 Whether the modern analytics do not furnish a strong *argumentum ad hominem* against the philomathematical infidels of these times?

Qu. 67 Whether it follows from the above-mentioned remarks, that accurate and just reasoning is the peculiar character of the present age? And whether the modern growth of infidelity can be ascribed to a distinction so truly valuable?

18.A2 Colin MacLaurin on rigorizing the fluxional calculus

The fluxion of the root A being supposed equal to a, the fluxion of the square AA will be equal to $2A \times a$.

Let the successive values of the root be $A-u$, A, $A+u$, and the corresponding values of the square be $AA - 2Au + uu$, AA, $AA + 2Au + uu$, which increase by the differences $2Au - uu$, $2Au + uu$, etc. and because those differences increase it follows from art. 704 that if the fluxion of A be represented by u, the fluxion of AA cannot be represented by a quantity that is greater than $2Au + uu$, or less than $2Au - uu$. This being premised, suppose, as in the proposition, that the fluxion A is equal to a; and if the fluxion of AA be not equal to $2Aa$, let it first be greater than $2Aa$ in any ratio, as that of $2A + o$ to $2A$, and consequently equal to $2Aa + oa$. Suppose now that u is any increment of A less than o; and because a is to u as $2Aa + oa$ to $2Au + ou$, it follows that if the fluxion of A should be represented by u, the fluxion of AA would be represented by $2Au + ou$, which is greater than $2Au + uu$. But it was shown from art. 704 that if the fluxion of A be represented by u the fluxion of AA cannot be represented by a quantity greater than $2Au + uu$. And these being contradictory, it follows that the fluxion of A being equal to a, the fluxion of AA cannot be greater than $2Aa$. If it can be less than $2Aa$, where the fluxion of A is supposed equal to a, let it be less in any ratio of $2A - o$ to $2A$, and therefore equal to $2Aa - oa$. Then because a is to u as $2Aa - oa$ is to $2Au - ou$, which is less than $2Au - uu$ (u being supposed less than o, as before) it follows that if the fluxion of A was represented by u, the fluxion of AA would be represented by a quantity less than $2Au - uu$, against what has been shown from art. 704. Therefore the fluxion of A being supposed equal to a, the fluxion of AA must be equal to $2Aa$.

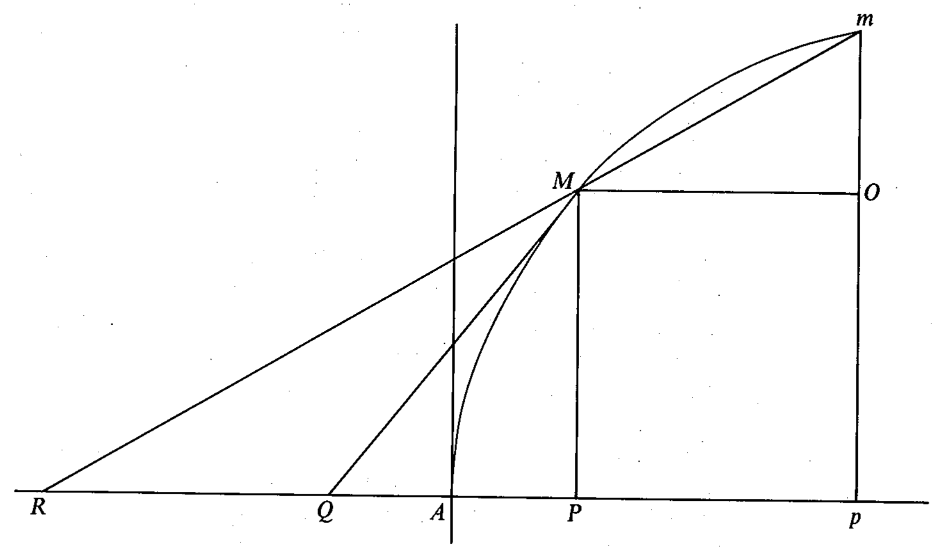
18.A3 D'Alembert on differentials

Newton started out from another principle; and one can say that the metaphysics of this great mathematician on the calculus of fluxions is very exact and illuminating, even though he allowed us only an imperfect glimpse of his thoughts.

He never considered the *differential* calculus as the study of infinitely small quantities, but as the method of first and ultimate ratios, that is to say, the method of

finding the limits of ratios. Thus this famous author has never differentiated quantities but only equations; in fact, every equation involves a relation between two variables and the differentiation of equations consists merely in finding the limit of the ratio of the finite differences of the two quantities contained in the equation. Let us illustrate this by an example which will yield the clearest idea as well as the most exact description of the method of the *differential* calculus.

Let AM be an ordinary parabola, the equation of which is $yy = ax$; here we assume that $AP = x$ and $PM = y$, and a is a parameter. Let us draw the tangent MQ to this parabola at the point M. Let us suppose that the problem is solved and let us take an ordinate pm at any finite distance from PM; furthermore, let us draw the line mMR



through the points M, m. It is evident, *first*, that the ratio MP/PQ of the ordinate to the subtangent is greater than the ratio MP/PR or mO/MO which is equal to it because of the similarity of the triangles MOm, MPR; *second*, that the closer the point m is to the point M, the closer will be the point R to the point Q, consequently the closer will be the ratio MP/PR or mO/MO to the ratio MP/PQ; finally, that the first of these ratios approaches the second one as closely as we please, since PR may differ as little as we please from PQ. Therefore, the ratio MP/PQ is the limit of the ratio of mO to OM. Thus, if we are able to represent the ratio mO/OM in algebraic form, then we shall have the algebraic expression of the ratio of MP to PQ and consequently the algebraic representation of the ratio of the ordinate to the subtangent, which will enable us to find this subtangent. Let now $MO = u$, $Om = z$; we shall have $ax = yy$, and $ax + au = yy + 2yz + zz$. Then in view of $ax = yy$ it follows that $au = 2yx + zz$ and $z/u = a/(2y + z)$.

This value $a/(2y + z)$ is, therefore, in general the ratio of mO to OM, wherever one may choose the point m. This ratio is always smaller than $a/2y$; but the smaller z is, the greater the ratio will be and, since one may choose z as small as one pleases, the ratio

$a/(2y + z)$ can be brought as close to the ratio $a/2y$ as we like. Consequently $a/2y$ is the limit of the ratio $a/(2y + z)$, that is to say, of the ratio mO/OM. Hence $a/2y$ is equal to the ratio MP/PQ, which we have found to be also the limit of the ratio of mO to OM, since two quantities that are the limits of the same quantity are necessarily equal to each other. To prove this, let X and Z be the limits of the same quantity Y. Then I say that $X = Z$; indeed, if they were to have the difference V, let $X = Z \pm V$: by hypothesis the quantity Y may approach X as closely as one may wish; that is to say, the difference between Y and X may be as small as one may wish. But, since Z differs from X by the quantity V, it follows that Y cannot approach Z closer than the quantity V and consequently Z would not be the limit of T, which is contrary to the hypothesis.

From this it follows that MP/PQ is equal to $a/2y$. Hence $PQ = 2yy/a = 2x$. Now, according to the method of the *differential* calculus, the ratio of MP to PQ is equal to that of dy to dx; and the equation $ax = yy$ yields $a\,dx = 2y\,dy$ and $dy/dx = a/2y$. So dy/dx is the limit of the ratio of z to u, and this limit is found by making $z = 0$ in the fraction $a/(2y + z)$.

But, one may say, is it not necessary also to make $z = 0$ and $u = 0$ in the fraction $z/u = a/(2y + z)$, which would yield $\frac{0}{0} = a/2y$? What does this mean? My answer is as follows. First, there is no absurdity involved; indeed $\frac{0}{0}$ may be equal to any quantity one may wish: thus it may be $= a/2y$. Secondly, although the limit of the ratio of z to u has been found when $z = 0$ and $u = 0$, this limit is in fact not the ratio of $z = 0$ to $u = 0$, because the latter one is not clearly defined; one does not know what is the ratio of two quantities that are both zero. This limit is the quantity to which the ratio z/u approaches more and more closely if we suppose z and u to be real and decreasing. Nothing is clearer than this; one may apply this idea to an infinity of other cases.

Following the method of differentiation (which opens the treatise on the quadrature of curves by the great mathematician Newton), instead of the equation $ax + au = yy + 2yz + zz$ we might write $ax + a0 = yy + 2y0 + 00$, thus, so to speak, considering z and u equal to zero; this would have yielded $\frac{0}{0} = a/2y$. What we have said above indicates both the advantage and the inconveniences of this notation: the advantage is that z, being equal to 0, disappears without any other assumption from the ratio $a/(2y + 0)$; the inconvenience is that the two terms of the ratio are supposed to be equal to zero, which at first glance does not present a very clear idea.

From all that has been said we see that the method of the *differential* calculus offers us exactly the same ratio that has been given by the preceding calculation. It will be the same with other more complicated examples. This should be sufficient to give beginners an understanding of the true metaphysics of the *differential* calculus. Once this is well understood, one will feel that the assumption made concerning infinitely small quantities serves only to abbreviate and simplify the reasoning; but that the *differential* calculus does not necessarily suppose the existence of those quantities; and that moreover this calculus merely consists in *algebraically determining the limit of a ratio, for which we already have the expression in terms of lines, and in equating those two expressions. This will provide us with one of the lines we are looking for.* This is perhaps the most precise and neatest possible definition of the *differential* calculus; but it can be understood only when one is well acquainted with this calculus, because often the true nature of a science can be understood only by those who have studied this science.