18.B3 Cauchy on two important theorems of the calculus

(a) The fundamental theorem of the calculus

If in the definite integral $\int_{x_0}^{X} f(x)\,dx$ we let one of the two limits of integration, for instance X, vary, the integral itself will vary with that quantity. And if the limit X, now variable, is replaced by x, we obtain as a result a new function of x, which will be what is called an integral taken from the *origin* $x = x_0$. Let

(1) $$\mathscr{F}(x) = \int_{x_0}^{x} f(x)\,dx$$

be that new function. We derive from [the formula $\int_{x_0}^{X} f(x)\,dx = (X - x_0)f[x_0 + \theta(X - x_0)]$ that]

(2) $$\mathscr{F}(x) = (x - x_0)f[x_0 + \theta(x - x_0)],\ \mathscr{F}(x_0) = 0,$$

θ being a [nonnegative] number less than one. Also, from [the formula $\int_{x_0}^{X} f(x)\,dx = \int_{x_0}^{\xi} f(x)\,dx + \int_{\xi}^{X} f(x)\,dx$, where $x_0 \leqslant \xi \leqslant X$],

$$\int_{x_0}^{x+\alpha} f(x)\,dx - \int_{x_0}^{x} f(x)\,dx = \int_{x}^{x+\alpha} f(x)\,dx = \alpha f(x + \theta\alpha),$$

or

(3) $$\mathscr{F}(x + \alpha) - \mathscr{F}(x) = \alpha f(x + \theta\alpha).$$

It follows from equations (2) and (3) that if the function $f(x)$ is finite and continuous in the neighbourhood of some particular value of the variable x, the new function $\mathscr{F}(x)$ will not only be finite but also continuous in the neighbourhood of that value, since an infinitely small increment of x will correspond to an infinitely small increment in $\mathscr{F}(x)$. Thus if the function $f(x)$ remains finite and continuous from $x = x_0$ to $x = X$, the same will hold for the function $\mathscr{F}(x)$. In addition, if both members of formula (3) are divided by α, we may conclude by passing to the limits that

$$\mathscr{F}'(x) = f(x).$$

Thus the integral (1) considered as a function of x has as its derivative the function $f(x)$ under the integral sign $\int$.

(b) The intermediate value theorem

Let $f(x)$ be a real function of the variable x, continuous with respect to that variable between $x = x_0, x = X$. If the two quantities $f(x_0), f(X)$ have opposite sign, the equation

$$f(x) = 0$$

can be satisfied by one or more real values of x between x_0 and X.

Proof Let x_0 be the smaller of the two quantities x_0, X. We will set $X - x_0 = h$ and we designate by m any integer greater than one. Because one of the two quantities $f(x_0)$, $f(X)$ is positive, the other negative, it follows that if we form the sequence

$$f(x_0),\ f(x_0 + h/m),\ f(x_0 + 2h/m), \ldots, f(X - h/m),\ f(X),$$

and compare successively the first term in that sequence with the second, the second with the third, the third with the fourth, etc., we necessarily will finish by finding once—or more than once—two consecutive terms that are of opposite sign.

Let $f(x_1), f(X')$ be two such terms, where x_1 is the smaller of the two corresponding values of x. Clearly we will have $x_0 < x_1 < X' < X$ and $X' - x_1 = h/m = 1/m(X - x_0)$. [Cauchy uses the sign $<$ for *less than or equal to*.] Having determined x_1 and X' in the way just described, similarly we can place between these two new values of x two other values x_2, X'', which, when substituted in $f(x)$, give results of opposite sign, and which satisfy the conditions $x_1 < x_2 < X'' < X'$ and $X'' - x_2 = 1/m(X' - x_1) = 1/m^2(X - x_0)$. Continuing thus, we obtain, first, a series of increasing values of x, that is,

(1) $$x_0, x_1, x_2, \ldots;$$

second, a series of decreasing values

(2) $$X, X', X'', \ldots$$

whose terms, since they exceed those of the first series by quantities equal, respectively, to the products

$$(X - x_0),\ (1/m)(X - x_0),\ (1/m^2)(X - x_0), \ldots,$$

ultimately will differ from the values in the first series by as little as desired. We must conclude from this that the general terms of the series (1) and (2) converge toward a common limit. Let a be that limit. Since the function $f(x)$ is continuous between $x = x_0$ and $x = X$, the general terms of the following series,

$$f(x_0), f(x_1), f(x_2), \ldots, f(X), f(X'), f(X''), \ldots$$

both converge toward the common limit $f(a)$; and since they always remain of opposite sign when they approach this limit, it is clear that the quantity $f(a)$, which must be finite, cannot differ from zero. Therefore, the equation

(3) $$f(x) = 0$$

will be satisfied if we give the variable x the particular value a, which lies between x_0 and X. In other words, $x = a$ will be a *root* of equation (3).

18.B4 J. V. Grabiner on the significance of Cauchy

What is our estimate of Cauchy's achievement? Cauchy's work established a new way of looking at the concepts of the calculus. As a result, the subject was transformed from a collection of powerful methods and useful results into a mathematical discipline based on clear definitions and rigorous proofs. His views were less intuitive than the old ones, but they provide a new set of interesting questions. His definition of limit and elaboration of the associated method of proof by the inequalities are the basis for modern theories of continuity, convergence, derivative, and the integral. And many of the important consequences of these theories—in the study of convergence, existence proofs for the solution of differential equations, and the properties of definite integrals—were pioneered by Cauchy himself.