In this section in particular, we will consider series of this last kind. If, for greater convenience, we set

$$z = r(\cos\theta + i\sin\theta), \tag{2}$$

where r denotes a real variable and θ a real arc, series (1) becomes

$$a_0 + a_1 r(\cos\theta + i\sin\theta) + a_2 r^2(\cos 2\theta + i\sin 2\theta) + \cdots + a_n r^n(\cos n\theta + i\sin n\theta) + \cdots. \tag{3}$$

Now . . . let A be the largest of the limits to which the nth root of the absolute value of a_n tends as n increases indefinitely.[7] The largest of the limits to which the nth root of the absolute value of the complex expression $a_n z^n = a_n r^n(\cos n\theta + i\sin n\theta)$ converges under the same assumption will be equivalent to the magnitude of the product Az; hence . . . the series (3) will be convergent or divergent depending on whether the product Az has a magnitude less than or greater than 1. From this remark, one immediately deduces the following proposition:

THEOREM. The series (3) is convergent for all values of z contained between the limits $z = -1/A$ and $z = +1/A$, and divergent for all values of z lying outside the same limits. In other words, the series (1) is convergent or divergent [according as] the absolute value of the complex expression z is less than or greater than $1/A$.

NOTES

1. A. L. Cauchy, *Oeuvres* (2), III, which reproduces his *Cours d'analyse de l'École Polytechnique*. I. *Analyse algebrique*. Selection 1a is from pp. 19, 43; 1*b* is from pp. 114-116; 1c is from pp. 239-240.
2. Here and elsewhere, Cauchy's "valeur numérique" has been translated as "absolute value" or "magnitude."
3. Cauchy writes "quantité infiniment petit." Note his clear specification of an infinitesimal as a *variable* quantity tending to zero, not a constant set equal to zero.
4. Here and elsewhere, Cauchy uses the phrase "entre des limites données" for "in a given interval."
5. Cauchy uses the word "série" for "sequence" and "series" alike; moreover, he puts a comma in (1) where today one writes a plus sign.
6. "Imaginaire" has been translated "complex" throughout this book. Cauchy uses x for z, z for r, and $\sqrt{-1}$ for i.
7. In modern notation, let $A = \limsup_{n\to\infty} |a_n|^{1/n}$.

103. From *Résumé des leçons . . . sur le calcul infinitésimal* (1823)*

(On the Derivative as a Limit[1])

AUGUSTIN-LOUIS CAUCHY

THIRD LESSON
DERIVATIVES OF FUNCTIONS OF ONE VARIABLE

When the function $y = f(x)$ is continuous between two given limits of the variable x, and one assigns a value between these limits to the variable, an infinitesimal increment Δx of the variable produces an infinitesimal increment in the function itself. Consequently, if we then set $\Delta x = h$,[2] the two terms of the *difference quotient* ["rapport aux différences"]

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} \tag{1}$$

will be infinitesimals. But whereas these terms tend to zero simultaneously, the ratio itself may converge to another limit, either positive or negative. This limit, when it exists, has a definite value for each particular value of x; but it varies with x. Thus, for example, if we take $f(x) = x^m$, m being a [positive] integer, the ratio of the infinitesimal differences will be

$$\frac{(x+h)^m - x^m}{h} = mx^{m-1} + \frac{m(m-1)}{1\cdot 2}x^{m-2}h + \cdots + h^{m-1},$$

and it will have for [its] limit the quantity mx^{m-1}, that is to say, a new function of the variable x. The same will hold generally; only the form of the new function which serves as the limit of the ratio $[f(x+h) - f(x)]/h$ will depend upon the form of the given function $y = f(x)$. In order to indicate this dependence, we give to the new function the name derivative ["fonction dérivée"] and we designate it, using a prime, by the notation y' or $f'(x)$.[3]

FOURTH LESSON
DIFFERENTIALS OF FUNCTIONS OF A SINGLE VARIABLE

Let $y' = f(x)$ remain a function of the independent variable x; let h be an infinitesimal and k a finite quantity. If we set $h = \alpha k$, α will also be an infinitesimal quantity, and we will have identically

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x+\alpha k) - f(x)}{\alpha k},$$

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whence one concludes that

$$\frac{f(x + \alpha k) - f(x)}{\alpha} = \frac{f(x + h) - f(x)}{h} k. \tag{1}$$

The limit toward which the left side of equation (1) converges as the variable α tends to zero, the quantity k remaining constant, is called the *differential* of the function $y = f(x)$. We indicate this differential by the symbol d, as follows:

$$dy \quad \text{or} \quad df(x).$$

It is easy to obtain its value when we know that of the derivative y' or $f'(x)$. Indeed, taking the limits of the two sides of equation (1), we shall find generally

$$df(x) = kf'(x). \tag{2}$$

In the special case where $f(x) = x$, equation (2) reduces to

$$dx = k. \tag{3}$$

Thus the differential of the independent variable x is just the finite constant k. Granting this, equation (2) becomes

$$df(x) = f'(x)dx \tag{4}$$

or, what amounts to the same thing,

$$dy = y'\,dx. \tag{5}$$

It follows from these last [equations] that the derivative $y' = f'(x)$ of any function $y = f(x)$ is precisely equal to dy/dx, that is, to the ratio of the differential of the function to that of the variable, or, if one wishes, to the coefficient by which the second differential must be multiplied in order to obtain the first. It is for this reason that we sometimes give to the derivative the name of *differential coefficient*.[4]

NOTES

1. A. L. Cauchy, *Résumé des leçons . . . sur le calcul infinitésimal* (Paris, 1823); *Oeuvres* (2), IV, 22ff, 27ff. Our translation has been adapted from the translation by Evelyn Walker (E. W.) in Smith, *Source Book*.
2. Cauchy uses i for h and h for k.
3. The phrase "fonction dérivée" and the notation $f'(x)$ were due to Lagrange.
4. After this Cauchy gives the rules for differentiating various elementary functions: algebraic, exponential, trigonometric, and inverse trigonometric. (E. W.)

104. From *Résumé des leçons . . . sur le calcul infinitésimal* (1823)*

AUGUSTIN-LOUIS CAUCHY

SEVENTH LESSON
THE FIRST RIGOROUS PROOF ABOUT DERIVATIVES

THEOREM. If the function $f(x)$ is continuous between the limits[1] $x=x_0$, $x=x$, and if we let A be the smallest, B the largest, value of the derivative $f'(x)$ in that interval, the ratio of the finite differences

$$[f(X) - f(x_0)]/X-x_0 \tag{4}$$

must be included[2] between A and B.

PROOF. Let δ, ϵ be two very small numbers; the first is chosen so that, for all numerical [i.e., absolute] values of i less than δ, and for any value of x included between the limits x_0, X, the ratio

$$f(x+i) - f(x)/i$$

will always be greater than $f'(x) - \epsilon$, [3]and less than $f'(x) + \epsilon$. If we interpose $n-1$ new values of the variable x between the limits x_0, X, that is

$$x_1, x_2, \ldots, x_{n-1},$$

so that the difference $X-x_0$ is divided into elements

$$x_1-x_0,\ x_2-x_1,\ \ldots,\ X-x_{n-1},$$

which all have the same sign and which have numerical values less than δ; then, since of the fractions

$$f(x_1)-f(x_0)/x_1-x_0,\ f(x_2)-f(x_1)/x_2-x_1,\ \ldots,\ f(X)-f(x_{n-1})/X-x_{n-1}. \tag{5}$$

the first will be included between the limits $f'(x_0) - \epsilon$, $f'(x_0) + \epsilon$, the second between the limits $f'(x_1) - \epsilon$, $f'(x_1) + \epsilon$, . . . , etc., each of the fractions will be greater than $A-\epsilon$, and less than $B+\epsilon$. Moreover, since the fractions (5) have denominators of the same sign, if we divide the sum of their numerators by the sum of their denominators, we obtain a mean fraction, that is, one included between the small-

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