Chapter VII
The Enlightenment (1720–1800)

Section A
Elaboration and Criticism of Mathematical Analysis

BROOK TAYLOR (1685–1731)

During the early Enlightenment, Brook Taylor was one of the few English mathematicians who could hold his own in disputes with Continental rivals, especially with John Bernoulli. The eldest son of John and Olivia Bart Taylor, he grew up in a comfortable family of the minor nobility. In 1701 Brook entered St. John's College, Cambridge (where he studied under John Mackin and John Keill). He received the LL. B. (1709) and LL. D. (1714). Elected a Fellow of the Royal Society in 1712, he sat on the committee that decided whether Newton or Leibniz deserved priority for the invention of the calculus. He was elected secretary of the Royal Society in 1714 but resigned the position in 1718 because of ill health and possibly from lack of interest in that confining task.

Taylor's most productive mathematical period dates from 1713 to 1719. He published two books on mathematics in 1715, entitled *Methodus Incrementorum Directa et Inversa* ("Direct and Indirect Methods of Incrementation") and *Linear Perspective*. During the period 1712 to 1724, he also wrote 13 articles for the *Philosophical Transactions* of the Royal Society and visited France on several occasions for social enrichment and to improve his health. In France, he met the mathematician Pierre Rémond de Montmort, and the two men subsequently corresponded with each other on the subjects of religion, infinite series, and probability. In this correspondence Taylor sometimes acted as an intermediary between Montmort and Abraham de Moivre in their studies of probability and suggested problems that needed to be solved.

After 1720, Taylor concentrated on art, philosophy, religion, music, and family matters. A brief rift with his morose father occurred when Taylor married a woman who had no fortune. After she died during childbirth in 1723, Taylor returned to his parent's home at Bifrons, Kent, and then remarried in 1725 (this time with his father's approval). He and his second wife, Sabetta Sambridge, moved to Bifrons estate when he inherited it in 1729. The next year Sabetta died during the birth of their daughter, Elizabeth, who survived. Taylor's delicate health rapidly worsened.

Taylor contributed to the early development of the calculus. He is best known for deriving the powerful formula for expanding a function into an infinite series that is now known as Taylor's theorem. He first explicitly stated it as Proposition VII of Theorem III in his book *Methodus Incrementorum*. In modern notation, it is

$$f(x+h) = f(x) + \frac{f'(x)h}{1!} + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \cdots + \frac{f^{(n)}(x)h^n}{n!} + \cdots$$

Conversant with the work of his predecessors, especially the British ones, he freely admitted a debt in deriving the theorem to Newton, Mackin, and Halley as well as to the German Kepler. Indeed he praised Newton, used his dot notation, and built upon Lemma V of Book III and Corollary II to Theorem III of the *Principia*. However, he was not so forthright about his indebtedness to contemporary Central European mathematicians. Even though he knew of Leibniz's pioneering work on finite differences and John Bernoulli's independent discovery of Taylor's theorem, which was given in *Acta eruditorum* (1694), he mentioned neither. Moreover, he did not worry over his lack of rigor in deriving the theorem nor did he seem to grasp the important role assigned to it by Lagrange, who in 1772 proclaimed it to be "the fundamental principle of differential calculus."

Because of its arithmetical exposition, the *Methodus* had little immediate influence in Britain where the primary attempts were to link the calculus to geometry or the physical notion of velocity. Peano later wanted to give John Bernoulli priority for the "Taylor series," but historians agree that Taylor deserves priority for integration by parts. Taylor and Bernoulli each claimed priority and sharply disagreed in other matters as well.

In the *Methodus* Taylor also established what is now called the calculus of finite differences as a new branch of higher analysis. With it he was able to reduce the motion of a vibrating elastic string to mechanical principles and to study the associated second order differential equation. He solved the equation $a^2\ddot{x} = \dot{s}y\dot{y}$, where $s = \sqrt{\dot{x}^2 + \dot{y}^2}$ and the differentiation is with respect to time, and gave $y + A \sin(x/a)$ as the form of the string at any time. In his overly concise book on linear perspective, he developed a theory of perspective in a formal and rigorous manner and presented the first general treatment of vanishing points and vanishing lines.

81. From *Methodus Incrementorum Directa et Inversa* (1715)*

(The Taylor Series)

BROOK TAYLOR[1]

Proposition VII. *Theorem* III. Let z and x be two variable quantities, of which z increases uniformly with given increments Δz.[2] Let $n\Delta z = v$, $v - \Delta z = \dot{v}$, $\dot{v} - \Delta z = \ddot{v}$, etc. Then I say that when z grows into $z + v$, then x grows into

$$z + \Delta x \,\frac{v}{1.\Delta z} + \Delta^2 x \,\frac{v\dot{v}}{1.2(\Delta z)^2} + \Delta^3 x \,\frac{v\dot{v}\ddot{v}}{1.2.3(\Delta z)^3} + \cdots,$$

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Fig. 1

DEMONSTRATION

x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$ etc.
$x + \Delta x$	$\Delta x + \Delta^2 x$	$\Delta^2 x + \Delta^3 x$	$\Delta^3 x + \Delta^4 x$	etc.
$x + 2\Delta x + \Delta^2 x$	$\Delta x + 2\Delta x + \Delta^3 x$	$\Delta^2 x + 2\Delta^3 x + \Delta^4 x$	etc.	
$x + 3\Delta x + 3\Delta^2 x + \Delta^3 x$	$\Delta x + 3\Delta^2 x + 3\Delta^2 x + \Delta^4 x$	etc.		
$x + 4\Delta x + 6\Delta^2 x + 4\Delta^3 x + \Delta^4 x$	etc.			

The successive values of x, collected by continued addition, are x, $x + 2\Delta x + \Delta^2 x$, $x + 3\Delta x + 3\Delta^2 x + \Delta^3 x$, etc., as we see from the operation expressed in the table. But the numerical coefficients of the terms x, Δx, $\Delta^2 x$, etc. for these values of x are formed in the same way as the coefficients of the corresponding terms in the binomial expansion [*in dignitate binomii*]. And if n is the exponent of the expansion [*dignitatis index*], then the coefficients (according to Newton's theorem) will be $1, \frac{n}{1}, \frac{n}{1}\frac{n-1}{2}, \frac{n}{1}\frac{n-1}{2}\frac{n-2}{3}$, etc. When, therefore, z grows into $z + n\Delta z$, that is, into $z + v$, then x will be equal to the series

$$x + \frac{n}{1}\Delta x + \frac{n}{1}\frac{n-1}{2}\Delta^2 x + \frac{n}{1}\frac{n-1}{2}\frac{n-2}{3}\Delta^3 x + \text{etc.}$$

But

$$\frac{n}{1} = \left(\frac{n\Delta z}{\Delta z}\right) = \frac{v}{\Delta z},\quad \frac{n-1}{2} = \left(\frac{n\Delta z - \Delta z}{2\Delta z}\right) = \frac{\dot{v}}{2\Delta z},\quad \frac{n-2}{3} = \left(\frac{n\Delta z - 2\Delta z}{3\Delta z}\right) = \frac{\ddot{v}}{3\Delta z},$$

etc. Hence in the time that z grows into $z + v$, x grows into[3]

$$x = x + \Delta x\,\frac{v}{1.\Delta z} + \Delta^2 x\,\frac{v\dot{v}}{1.2(\Delta z)^2} + \Delta^3 x\,\frac{v\dot{v}\ddot{v}}{1.2.3(\Delta z)^3} + \text{etc.}$$

Corollary I. If the Δz, Δx, $\Delta^2 x$, $\Delta^3 x$ remain the same, but the sign of v is changed so that z decreases and becomes $z - v$, then x decreases at the same time and becomes

$$x - \Delta x\,\frac{v}{1.\Delta z} - \Delta^2 x\,\frac{v\dot{v}}{1.2(\Delta z)^2} - \Delta^3 x\,\frac{v\dot{v}\ddot{v}}{1.2.3(\Delta z)^3} - \text{etc.}$$

or[4]

$$x - \Delta x\,\frac{v}{1.\Delta z} + \Delta^2 x\,\frac{v v_{\prime}}{1.2(\Delta z)^2} - \Delta^3 x\,\frac{v v_{\prime} v_{\prime\prime}}{1.2.3(\Delta z)^3} + \text{etc.}$$

with v, $\ddot{v}$, etc. converted into $-v_{\prime}$, $-v_{\prime\prime}$, etc.

Corollary II. If we substitute for evanescent increments the fluxions proportional to them, then all $\ddot{v}$, $\dot{v}$, v, $v_{\prime}$, $v_{\prime\prime}$ become equal. When z flows uniformly into $z + v$, x becomes[5]

$$x + \dot{x}\,\frac{v}{1.\dot{z}} + \ddot{x}\,\frac{v^2}{1.2\dot{z}^2} + \dddot{x}\,\frac{v^3}{1.2.3\dot{z}^3} + \text{etc.,}$$

or with v changing its sign, when z decreases to $z - v$, x becomes

$$x - \dot{x}\,\frac{v}{1.\dot{z}} + \ddot{x}\,\frac{v^2}{1.2\dot{z}^2} - \dddot{x}\,\frac{v^3}{1.2.3\dot{z}^3} + \text{etc.,}$$

[*Text omitted.*]

Here follows one of Taylor's applications of the theorem.

Proposition VIII. *Problem* V. Given an equation which contains, apart from a uniformly increasing z, a certain number of other variables x. To find the value of x from given z by a series of an infinite number of terms.

Find all increments, to infinity, of the proposed equation by means of Proposition I. If $\Delta^n x$ be the infinite increment of x in the proposed equation, then by means of these equations will be given all increments $\Delta^n x$ and those with higher n expressed by means of increments of lower n. Let a, c, c_1, c_2, c_3, etc. be certain arbitrary values corresponding to z and x, Δx, $\Delta^2 x$, $\Delta^3 x$, etc.; then by means of these equations all terms c_n, c_{n+1}, and the following can be expressed in terms of the terms preceding c_n. Hence if we write $a + v$ for z, then x will be given by means of

$$x = c + c_1\,\frac{v}{1.\Delta z} + c_2\,\frac{v\dot{v}}{1.2(\Delta z)^2} + c_3\,\frac{v\dot{v}\ddot{v}}{1.2.3(\Delta z)^3} + \text{etc.}$$

(according to Proposition VII). Here the coefficients c, c_1, c_2, etc. of the terms whose number is n are given by the same number of conditions imposed on the problem.[6]

NOTES

1. Taylor used a complicated notation with dots and primes (*lineolae*) used as superscripts and subscripts, and the primes in both the *accent grave* and *accent aigu* position. We have kept his notation, except that instead of the increments $z_{\prime}$, $z_{\prime\prime}$, etc. we have written Δz, $\Delta^2 z$, etc., and for the v with subscript *accents aigus* we have written $v_{\prime}$, $v_{\prime\prime}$, ... Taylor's notation also has its advantage. In his notation x'', x, x', $\dot{x}$, $\ddot{x}$ represent a sequence of functions of which each is the fluxion of the previous one; whereby Taylor remarks that the *lineolae* in x', x'' can be regarded as negative dots—an anticipation, if we like, of our modern operational notation D^{-2}, D^{-1}, D^0, D^1, D^2, and for the same purpose.

2. Since z flows uniformly, Δz is constant, so that $\Delta^2 z$, $\Delta^3 z$, etc. are all zero. Here $\Delta^2 z$ is the increment of Δz, $\Delta^3 z$ that of $\Delta^2 z$, etc.

3. This is Newton's well-known interpolation formula (*Principia*, Book III, Lemma 5); see also Newton, *Methodus differentialis* (London, 1711); *James Gregory tercentenary memorial volume*, ed. H. W. Turnbull (Bell, London, 1939), 119; H. W. Turnbull, *The mathematical discoveries of Newton* (Blackie, Glasgow, 1945), 46.

4. The $v_{\prime\prime}$, $v_{\prime}$, v, $\dot{v}$, $\ddot{v}$ form a sequence of increments, so that $v_{\prime} - \Delta z = v$, $v_{\prime\prime} - \Delta z = v_{\prime}$, or $v_{\prime} = (n+1)\Delta z$, $v_{\prime\prime} = (n+2)\Delta z$.

5. This is the classical Taylor series, since in the Leibniz notation $\dot{x}/\dot{z} = dx/dz$, $x/(\dot{z})^2 = d^2x/dz^2$, etc. Taylor therefore obtained his series from Newton's interpolation formula by taking $\Delta x = 0$, $n = \infty$. Felix Klein has called Taylor's step "a transition to the limit of extraordinary audacity"; see *Elementary mathematics from an advanced standpoint*, trans. E. R. Hedrick and C. A. Noble, I (Dover, New York, 1924), 233. Although we shall not belittle this statement we must also take into account that Taylor's theorem had been "in the air" ever since James Gregory had it in a manuscript of 1671 (*Gregory tercentenary memorial volume*, pp. 123, 173, 356). See also A. Pringsheim, "Zur Geschichte des Taylorschen Lehrsatzes," *Bibliotheca mathematica (3) 1* (1900), 433-479; G. Eneström, "Zur Vorgeschichte der Entdeckung des Taylorschen Lehrsatzes," *ibid., 12* (1911-12), 333-336.

6. This proposition and several others give information on the number of arbitrary constants in difference and differential equations. Taylor, on page 27 of his book, shows how the differential equation $\ddot{x} - \dot{x}z - 2x = 0$ can be solved by means of $x = A + Bz + Cz^2 + Dz^3 +$