Two Famous Irrational Numbers

First we'll prove that e is irrational. Notice at least that $2 < e < 3$, so e is not an integer. Like all good irrational proofs, we begin by assuming $e = \frac{p}{q}$ with $q \geq 2$ (here is where we use that e is not an integer). What definition are we using for e? This one:

$$\frac{p}{q} = e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Now let's multiply by $q!$. Notice the left-hand-side is then an integer

$$p(q-1)! = q! + q! + q \cdots 4 \cdot 3 + q \cdots 5 \cdot 4 + \cdots + q(q-1) + q + 1 + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots$$

Let's focus on the part after $\cdots + q + 1$ which is not obviously an integer. Recall $q \geq 2$. So, $q + 1 \geq 3$ and all others as well.

$$\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \leq \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

This is a geometric series, which converges to the sum of $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

So, the left hand side is an integer, and the right hand side is definitely not. This is a contradiction.

This idea generalises nicely using power series. The same reasoning will prove that $e^{\frac{p}{q}}, \sin \frac{p}{q}, \cos \frac{p}{q}$ are all irrational.

So, now the real one. Here is the famous result that π is irrational. As should be expected we begin by supposing $\pi = \frac{p}{q}$. But that's where it stops being predictable. And here we go to the interesting part. Consider this function $f_n = \frac{x^n(p-qx)^n}{n!}$. Because $\pi = \frac{p}{q}$, there are several useful ways we can rewrite this function:

$$f_n = \frac{x^n(p-qx)^n}{n!} = \frac{x^n(q\pi - qx)^n}{n!} = \frac{x^n q^n(\pi - x)^n}{n!} = \frac{x^n(p - \frac{p}{\pi}x)^n}{n!} = \frac{x^n p^n(1 - \frac{1}{\pi}x)^n}{n!}$$

We'll have opportunity to think about several of these versions, but it's all the same function. As always in mathematics - the more choices you have, the more we can do with it.

Now, let's notice some things.

- $n!f$ is a polynomial with integer coefficients.
- $n!f(\pi - x) = (\pi - x)^n(p - q(\pi - x))^n = \left(\frac{p}{q} - x\right)^n \left(p - q\left(\frac{p}{q} - x\right)\right)^n = \left(\frac{p-qx}{q}\right)^n (p - p + qx)^n = (p - qx)^n \left(\frac{qx}{q}\right)^n = (p - qx)^n x^n = f(x)n!$
- So, $f^{(n)}(x) = (-1)^n f^{(n)}(\pi - x)$
- If $0 \leq x \leq \pi$ then $0 \leq f(x) \leq \frac{\pi^n p^n}{n!}$ using the last version of f
- For $j < n$, $f^{(j)}(0) = f^{(j)}(\pi) = 0$
- For $j \geq n$, $f^{(j)}(0) = (-1)^j f^{(j)}(\pi) \in \mathbb{Z}$

Now let $F_n(x) = f_n(x) - f_n''(x) + f_n^{(4)}(x) - f_n^{(6)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)$ (notice that's the end of derivatives because f_n is a polynomial of degree $2n$). From our last two facts about f, notice $F(0) = F(\pi) \in \mathbb{Z}$ and $F + F'' = f$.

All that is basically our set-up. And now for the finish ....

And now we use some different ideas to finish. We're going to be focused on $f \sin x$. Consider $F'(x)\sin x - F(x)\cos x$ Now notice that

$$(F'(x)\sin x - F(x)\cos x)' = F''(x)\sin x + F'(x)\cos x - F'(x)\cos x + F(x)\sin x = (F''(x) + F(x))\sin x = f \sin x$$

And this is a good set up for integrating $f \sin x$. So, now we see

$$\int_0^\pi f \sin x dx = \int_0^\pi (F' \sin x - F \cos x)' dx = [F'(\pi)\sin\pi - F(\pi)\cos\pi] - [F'(0)\sin 0 - F(0)\cos 0] = 0 + F(\pi) - 0 + F(0) \in \mathbb{Z}$$

But $\int_0^\pi f_n(x)\sin x dx \leq \int_0^\pi \frac{\pi^n p^n}{n!} dx \leq \frac{\pi^{n+1} p^n}{n!}$ Now, for the first time of all this, remember that n is a variable. Eventually, because factorial grows faster than any exponential (in this case because $n > \pi p$) there is a n such that $\frac{\pi^{n+1} p^n}{n!} < 1$. But we have both that this integral is an integer, and also that it is strictly between 0 and 1. This is our contradiction, and finally, we have our goal π is, as we always say, irrational. And we probably would've never thought to do any of this, but now we know it is true, and we know that we have seen it - and we're not just trusting authority.