Mathematics 228 PS4 Solutions

§7.1 38

In the first case

	A	a
A	0.28	0.42
a	0.12	0.18

Pr(Aa) = 0.54

in the second case

	A	a
A	0.35	0.15
a	0.15	0.35

Pr(Aa) = 0.3

§7.2 28

After elimination the 0, 1, 2 probabilities are $\frac{4}{9}$, $\frac{3}{9}$, and $\frac{2}{9}$ respectively. We have the following joint distribution (S = 1 means stained)

	1	0
0	$\frac{19}{45}$	$\frac{1}{45}$
1	$\frac{3}{10}$	$\frac{1}{30}$
2	$\frac{8}{45}$	$\frac{2}{45}$

$E(A) = \frac{7}{9}$

$E(S) = \frac{9}{10}$

$E(AS) = \frac{59}{90}$

So, $\mathrm{Cov}(A,S) = -\frac{2}{45}$. The negative makes sense, as it's harder to stain as the age increases, so as A increases, S tends to decrease.

§7.2 32

The joint distribution for S,P is given by

	8	9	5
10	.7	0	0
14	0	.2	0
18	0	0	.1

E(S) = 11.6
E(P) = 7.9
E(SP) = 90.2 so Cov(S,P) = −1.44
$E(S^2)$ = 141.6 $E(P^2)$ = 63.5, so Var(S) = 7.04 and Var(P) = 1.09
therefore $\rho(S,P) = -0.5198319$
This is not −1 because the bird also eats, unlike in the book example.

§7.3 36 (extra credit)

The expected revenue of the first is $1000(.6+.2) = $800
The second is $1000(.2+.4) = $600
The third is $500(.3+1.2) = $750
The total is therefore $2150.

The variance for the first is ($1000)^2(0.36) = $360,000
The second is ($1000)^2(0.64) = $640,000
The third is ($500)^2(0.45) = $112,500
If independent the total variance is $1,112,500
The total standard deviation is then $3335, which is more than the total mean. This is not very predictable business.

§7.4 52

We have $\binom{5}{3}$ = 10 ways to distribute medication. One is correct, so the probability of being correct is 0.1.

§7.4 58

This follows binomial – both alleles are independent and identical, so the probabilities are $1(.8)^2(.2)^0 = 0.64$ AA, $2(.8)^1(.2)^1 = 0.32$ Aa, and $1(.8)^0(.2)^2 = 0.04$.

§7.5 (37, 39, 40) as one question

We expect $40(.2)^t$ remaining t minutes later

The variance is $40(.2)^t(1 - (.2)^t)$.

We take the derivative, notice that the maximum does not occur at 0, and find where the derivative is zero to be the maximum, t = .8126795911 minutes, but this is discrete, not continuous, so we check and find that 1 is larger than 0, so the maximum is at one minute.

The coefficient of variation is $\frac{\sqrt{40(.2^t - .04^t)}}{40(.2)^t}$ a graph reveals that this function is increasing.

§7.6 (59-64) as one question

Following the text on pp. 652–3, we have

$\frac{dP}{dt} = -tP$

separate

$\int \frac{dP}{P} = \int -t\,dt$

$\ln(P) = -\frac{1}{2}t^2 + C$

$P = e^{-\frac{1}{2}t^2}$ is the probability surviving to time t. Graph it on $[0,4]\times[0,1]$.

The cdf is $1 - P = 1 - e^{-\frac{1}{2}t^2}$ Differentiate to find the pdf $te^{-\frac{1}{2}t^2}$

P(1) = .6065306597 is the probability surviving to age 1

P(2)/P(1) = .2231301601 is the probability surviving to age 2 conditional on surviving to age 1. This is lower than the previous answer because death rate increases with age. This would not be the case for a constant death rate.

§7.7 58

$\sum_{k=0}^{\infty} \frac{1}{k!}(1-q)^k e^{-\lambda t}(\lambda t)^k$

k corresponds to how many times the organism has been attacked. Notice the $(1-q)^k$ corresponds to attacks failing. Notice also that this does work for k = 0 as desired.

§7.8 36

The expected value of the annual change is −3.5. The variance is 28.45. So, the expected value of the 10 year average is also −3.5, but the variance is 2.845. So, the pdf is $\frac{1}{\sqrt{2\pi(2.845)}} e^{-(x+3.5)^2/2(2.845)}$. The maximum average possible is 2, the minimum possible is −10. This should gives us a clue for where to plot. [−2,10]×[0,0.3] looks good. Not much above 0, maybe 5%? How do we feel about the approximation? Looks pretty ok, but could argue that it isn't.

§7.9 50

Using the binomial distribution the mean is 4 and the variance is 3.992. We may then use the normal approximation. The appropriate pdf is $\frac{1}{\sqrt{2\pi 3.992}} e^{-(x-4)^2/2(3.992)}$, which we integrate from .5 to 1.5 to get 0.065513

If you use the Poisson approximation to the binomial (as Adler may suggest) the variance is 4 instead of 3.992 yielding 0.0655906

§8.1 44

We can do this either using Poisson (as the book does, and is hinted by the use of λ) or using binomial. I'll do Poisson first:

The probability of 12 out of 200 nonsynonymous is given by $\frac{e^{-\lambda 200}(200\lambda)^{12}}{12!}$ and the probability of 15 out of 100 synonymous is given by $\frac{e^{-3\lambda 100}(100\cdot 3\lambda)^{15}}{15!}$. So the probability both happen is $L(\lambda) = \left(\frac{e^{-\lambda 200}(200\lambda)^{12}}{12!}\right)\left(\frac{e^{-3\lambda 100}(100\cdot 3\lambda)^{15}}{15!}\right)$

$= \frac{e^{-500\lambda}200^{12}300^{15}\lambda^{27}}{15!12!}$. The constants don't affect our answer, so we write them hereafter

$L'(\lambda) = -500e^{-500\lambda}\lambda^{27} + 27e^{-500\lambda}\lambda^{26} = \lambda^{26}e^{-500\lambda}(27 - 500\lambda) = 0$

$L(0) = 0$, $L\left(\frac{27}{500}\right) = 0.000000116622 \neq 0$, so $\frac{27}{500} = 0.054$ is the mle. Seems to fit the data.

Now redoing with binomial. The probability of 12 out of 200 nonsynonymous is given by $\binom{200}{12}\lambda^{12}(1-\lambda)^{188}$ and the probability of 15 out of 100 synonymous is given by $\binom{100}{15}(3\lambda)^{15}(1-3\lambda)^{85}$. So the probability both happen is $L(\lambda) = \binom{200}{12}\lambda^{12}(1-\lambda)^{188}\binom{100}{15}(3\lambda)^{15}(1-3\lambda)^{85}$

$= \binom{200}{12}\binom{100}{15}\lambda^{27}(1-\lambda)^{188}3^{15}(1-3\lambda)^{85}$. The constants don't affect our answer, so we don't write them hereafter. Use the product rule twice to

$L'(\lambda) = -\lambda^{26}(1 - \lambda)^{187}(1 - 3\lambda)^{84}(900\lambda^2 - 551\lambda + 27) = 0$

$L(1) = L(0) = L(1/3) = 0$, 0.55850764 is unrealistic, as it produces $\lambda_s < 0$ Since $L(0.053714573208) = 0.0116009 > 0$, 0.053714573208 is the mle. Close to the previous, a little lower.

§8.3 (33-36) as one question

For each single measurement, we have $p(1 - p)$ as a variance, in this case the variance is $\frac{1}{4}$.

sample	probability	$\bar{x}$	msd	sample variance
(0,0)	$\frac{1}{4}$	0	0	0
(1,0)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
(1,1)	$\frac{1}{4}$	0	0	0

See that the expected value of msd = $\frac{1}{8}$, whereas the expected value of the sample variance is $\frac{1}{4}$, which is the actual variance. This is an example of why we use the sample variance with $n - 1$ instead of n.

§8.4 30

The survival time is exponentially distributed with a mean of 100 h. Because the mean of an exponential distribution is $\frac{1}{\lambda}$, we have an exponential distribution with parameter 0.01. We seek a survival time, s, such that the probability of it occurring is less than or equal to 0.05, i.e. such that $\int_0^s 0.01e^{-0.01t}dt \leq 0.05$

This yields $1 - e^{-0.01s} \leq 0.05$

solving for s produces $s \leq 5.129$ h. It must be pretty small to be significantly different from 100. It's worth recalling that the standard deviation of the exponential distribution with mean 100 is 100, so it's a pretty spread out distribution. Also that the mode value is 0, so getting low values isn't entirely uncommon.