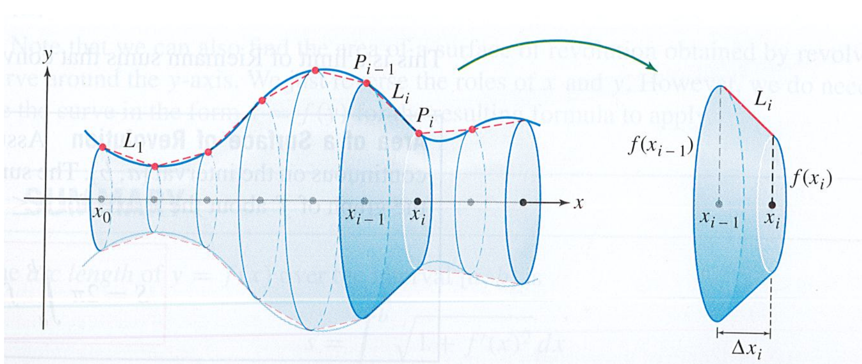
222 §1.4 Arc Length and Surface Area

Let's start with takeaways from Lab 14. It the long page between questions 3 and 4, in the middle the authors say "Therefore the arc length equals ...". (1) What is the derived integral formula for arc length of a function f from $x = a$ to $x = b$? The lab then goes on to say "in practice it is difficult (or impossible) to" exactly compute these integrals. That does *not* mean that setting them up is useless. You know some ways to approximate integrals using rectangles from Calculus I. (2) What were they? We will discuss several more at the end of September in Lab 18 and section 2.6. Using these techniques, once we have anything set up as an integral we can approximate it as closely as we want for all practical purposes. Because the integrals are difficult (actually they *are* usually impossible) to compute we will mostly only set up integrals today. As a consequence, we will be able to set up far more, because we're only setting up.

The funny truth is that you see the same examples of this over and over because there are only a few that can be computed exactly, and they are specially constructed to work. Here are some. (3) Set up the integral to compute the arc length of the graph of $f(x) = 2x^{3/2}$ over the interval $[0, 7]$. There's a convenient square that cancels a square root (I told you these were set up to work out). (4) Let $u =$ the inside of the square root to (5) complete this integral computation.

Here's another of the few examples of arc length that is computable: calculate the length of the graph of the function $f(x) = \frac{e^x+e^{-x}}{2}$ over the interval $[1, \ln 8]$. It's pretty clear these are artificially created examples. (6) Set up the integral. When you square the derivative and find a common denominator with the 1 in the formula, magically (as if someone went out of their way to make it happen), the result of the numerator is something that can be factored as a perfect square again. And then we can take the square root. (7) Do all that, then (8) complete the integral.

Really there's been nothing new so far, this is just more examples from what we saw in lab. Here's our one new topic for today. Using the same methods for arc length we can also find a formula for surface area. This combines our ideas from volumes and arc length. So, from arc length we approximate arc length by splitting our curve into small line segments. Then if we rotate those small line segments around an axis we can approximate surface area. Look at the following picture to see what is happening.



Each of the little segments forms a piece of surface area as illustrated on the right. This shape is called a *frustum.* (9) It is made by cutting the top off of what shape? (10) Sometimes the segment will be parallel to the axis, in that case, what more familiar shape do we get? (11)

We saw it recently, what is the surface area of a cylinder? The formula for frusta (that's plural of frustum, we've got some fancy words today) is similar. For a cylinder you have a radius. For a frustum you have a top and bottom radius and they average together to balance out. For a frustum with radii R and r and slant height s, the surface area is $2\pi\left(\frac{r+R}{2}\right)s$. In our context, forming a frustum by rotating a segment around an axis, the slant height of the frustum plays the role of the height of the cylinder. That is the same $\sqrt{1+[f'(x)]^2}$ from the arc length. The radii are the two different function values at the two endpoints, but as we take limits those two ends become identical, so the surface area of *our* frustum is $2\pi f(x)\sqrt{1+[f'(x)]^2}\Delta x$. Then pulling that into an integral, we find the following formula for the surface area of a surface produced by rotating the function $y = f(x)$ over the interval $[a, b]$ rotated around the x-axis:

$$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}dx$$

Let's set up some examples. Let $f(x) = x^3$. For $1 \leq x \leq 2$, we will rotate the graph of f around the x-axis. (12) What integral gives the surface area? (13) We're not going to finish this one, but what would the next step be?

Probably the most famous sophisticated surface area is that of a sphere of radius r. We hope you recall a formula, (14) what is it? We expect you have no idea where that comes from. One way to get it is from calculus. Let's think about setting it up, at least. (15) What shape do you need to spin around the x-axis to get a sphere? (16) What is the function whose graph gives that shape? (17) What is the range of values? (18) Put all of this into the integral for surface area. (19) Surprisingly we know all we need to complete this integral, but the algebra might be a tad scary. We'll see how far we want to take it.

What if we want to rotate around the y-axis instead? (20) What one simple change would we need to make in our formula? As an example, we will think about rotating $y = x^4$ over the interval $[1, 2]$ around the y-axis. We need to switch our variables, in doing so (21) what does $x =$ in terms of y? (22) When we work in terms of y, what are the new limits? (23) Putting this into our integral formula for surface area, what integral gives the surface area of this surface? We definitely will *not* evaluate this integral. This is one for numerical integration. There's a motivation for that topic at the end of September.