222 §1.2 Volumes by Slicing

Today we will go into another dimension ... the third dimension. The basic idea of integration, going back to your introduction in your first calculus course, is that it is a way way to add up infinitely many thin slices. When you drew your pictures then, they started as rectangles and became more thin, and you added up thin slices to get area. Today, we will use the same idea to add up infinitely thin slices of *area* to get *volume*, the third dimension!

I want to start with the simplest most concrete example I can think of, and then we can generalise. For the first example, I'm thinking of a pyramid. Like the pyramids in Egypt, but less big.

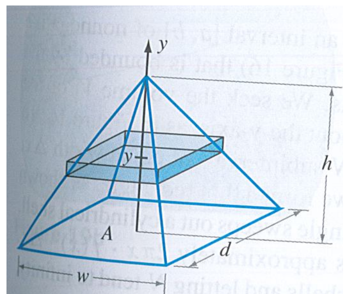

Figure 13

(1) what is the shape on the base? To start with my first example, I am going to think of a pyramid with its point down (because it makes the mathematics easier to see, although we could just use coordinate going down). It is 10 units high and 10 units across at the base. This makes it very easy, because at height h the side length is also h. If we slice the pyramid horizontally (parallel to the ground), (2) what shape do you see, and what is its area at height h (it is in terms of h, it is 0 at the bottom, when $h = 0$)? So, the main idea here is that we want to add up the cross sections from the bottom to the top. We will do this using an integral. We have the pieces, the bottom height is zero the top is ten, and we're integrating with respect to h which is changing. And you found the cross section area, so (3) what is our integral? If you do the numerical integration you should get 1000/3 as an answer, which fits if you know the volume of a pyramid formula.

Let's generalise this in a couple somewhat small ways before we move on. instead of having the height always equal the sidelength, what if the top side length is s and the height is a (for altitude)? The bounds on the integral need to change. That should be clear. But, the side-length at height h is trickier. At $h = 0$ the width is 0. At $h = a$ the width is s. (4) Using slope, what is the width at generic height h? (5) And so, what is the cross-sectional area at generic height h? (6) Hence, what integral gives the volume of a pyramid with sidelength s and height a? (7) Complete this integral and compare it to a formula for the volume of a pyramid. We hope it looks familiar.

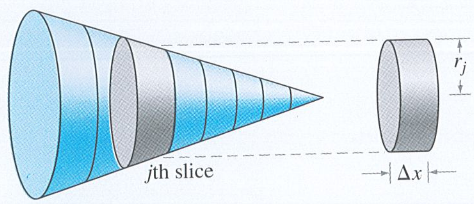

Figure 2
A disk is used to approximate a slice of the cone.

What if we do the same thing with a cone? It should be about the same. Let's keep the height a, and use a top radius of r (think of an ice-cream cone, point down). The important difference is that the cross-section shape changes. (8) If you slice the cone horizontally, what shape do you get? (9) Basically similar to the above, what is the radius at height h? (10) From this, what is the cross-sectional area at height h? (11) Using this, what integral gives the volume of a cone with radius r and height a? (12) Complete this integral and compare it to a formula for the volume of a cone. This also should be familiar.

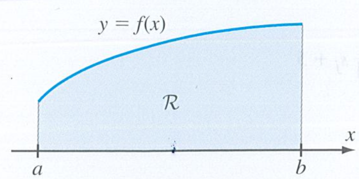

Figure 4

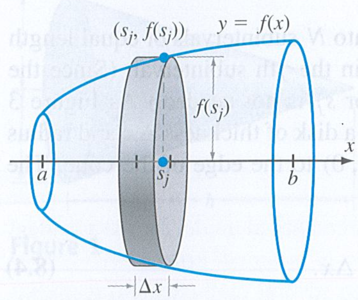

Figure 5

The cone is a special case of a collection of objects can be made by spinning around an axis. We will consider this for the rest of today and the next day. Instead of rotating a right triangle around its height (which I hope you see makes a cone) we can rotate the area of any other region around an axis. If the region is bounded by the axis on one side (as it is with the right triangle rotating around its height), this is rather simple. The cross sections are still circles. Here's a starting example: Calculate the volume enclosed when the graph of $x = \frac{1}{2}y^3$ for y between 2 and 4 is rotated around the y-axis. The cross sections are still circles, the integral is now with respect to y instead of h, but it's very similar. So ... (13) What is the integral for computing the volume of this solid?

We can, naturally, do something similar around the x-axis. As an example, what is the volume produced when rotating the region between the x-axis and the curve $y = \sqrt{x}$ for x between 0 and 4? Again, setting up the integral should be very similar. (14) What is the integral needed for this example?

In the above examples, the region is bounded by the axis on one side. But, with a little more work and thought, we can handle examples where the region is not bounded by the axis of rotation. Here's one for practice ... consider the region of the xy-plane bounded above by $y = 8\sqrt{x}$ and $y = x^2$. Calculate the volume when this region is rotated around the x-axis. (15) Start by making the graph of the region. Much of the setup to start is similar to what we did for finding the area. (16) Where do the curves intersect? That will give your limits of integration. Now, the visual challenge, think of what is produced when rotating that region around the x-axis. It's a solid that has a whole on the inside, opening to the right. Now imagine cutting a vertical slice through this object. The most important question as we wind this up is (17) what shape is produced when cutting vertically? You might not know a technical word for it, does it remind you of anything you've seen? Technically it's an annulus, a circle with a central circle removed. You might call it a "washer" (not like for clothes or dishes, but the kind that goes around bolts). (18) Given what I've said, how do you find the area of an annulus (think of a two step process)? So, in order to do so, you'll need an inner radius and outer radius. (19) What is the distance from the x-axis to the inner curve, that's the inner radius? (20) What is the distance from the the x-axis to the outer curve, that's the outer radius? (21) Putting those together, what is the area of the cross section? And finally, (22) what integral gives the volume for this solid?

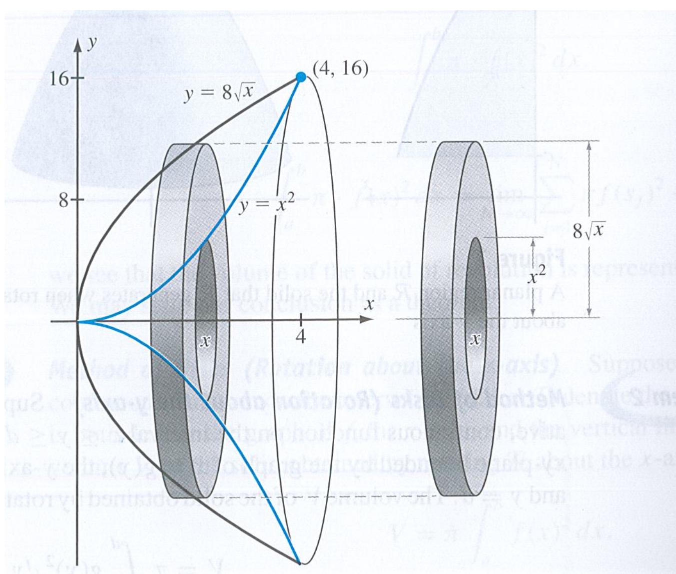

Figure 9

We will work more with volumes tomorrow. That seems like plenty for today.