In Exercises 37–40, use the Method of Disks to calculate the volume obtained by rotating the planar region $\mathcal{R}$ about the y-axis.

37. $\mathcal{R}$ is the region between the curves $y = \exp(x)$, the y-axis, and $y = e$.

38. $\mathcal{R}$ is the region between the curve $y = \sin(x^2)$ and the lines $x = 0$ and $y = 1$.

39. $\mathcal{R}$ is the region between the curve $y = x^2/(1 - x^2)$ and the lines $x = 0$ and $y = 1/3$.

40. $\mathcal{R}$ is the region in the first quadrant bounded by the coordinate axes and the curve $x = (1 - y^2)^{3/4}$.

In Exercises 41–46, use the Method of Cylindrical Shells to calculate the volume obtained by rotating the planar region $\mathcal{R}$ about the given line ℓ.

41. $\mathcal{R}$ is the region between the curves $y = x^2 - 4x - 5$ and $y = -x^2 + 4x - 3$; ℓ is the line $x = -3$.

42. $\mathcal{R}$ is the region between the graphs of $y = \exp(x)$, $y = x\exp(x)$, and the y-axis; ℓ is the line $x = 0$.

43. $\mathcal{R}$ is the region between the curves $y = x^2 - x - 5$ and $y = -x^2 + x + 7$; ℓ is the line $x = 6$.

44. $\mathcal{R}$ is the region between the curves $x = y^3$ and $x = -y^2$; ℓ is the line $y = 3$.

45. $\mathcal{R}$ is the region between the curves $x = y^4$ and $x = y^{1/2}$; ℓ is the line $y = -1$.

46. $\mathcal{R}$ is the region between the curves $y = 2\sin(x)$, $\pi/6 \le x \le 5\pi/6$, and $y = 1$; ℓ is the line $y = -1$.

In Exercises 47–54, choose disks, washers, or cylindrical shells, whichever is best, to calculate the volume of the solid obtained when the region $\mathcal{R}$ is rotated about the line ℓ.

47. $\mathcal{R}$ is the region between $y = -x^2 + 6$ and $y = -x$; ℓ is the line $x = -4$.

48. $\mathcal{R}$ is the region between the curves $y = x^2$ and $y = x^{1/2}$; ℓ is the y-axis.

49. $\mathcal{R}$ is the region between the curve $y = \cos(x)$ and the x-axis, $\pi/2 \le x \le 3\pi/2$; ℓ is the line $x = 3\pi$.

50. $\mathcal{R}$ is the region between the curve $x = \tan(y)$ and the y-axis, $0 \le y \le \pi/4$; ℓ is the line $x = 4$.

51. $\mathcal{R}$ is the region bounded by the curve $x = \sin(y)$, the y-axis, and the line $y = \pi/2$; ℓ is the line $y = 4$.

52. $\mathcal{R}$ is the region between the curves $y = \sin(x)$ and $y = \cos(x)$, $0 \le x \le \pi/4$; ℓ is the line $y = 4$.

53. $\mathcal{R}$ is the region between the curves $y = x^3 + x$, $y = 0$, and $x = 1$; ℓ is the line $x = 0$.

54. $\mathcal{R}$ is the region between the curves $y = 1/(1 + x^2)$, $y = 1/2$, and $x = 0$; ℓ is the line $x = 0$.

55. Suppose $R > r > 0$. Calculate the volume of the solid obtained when the disk $\{(x, y) : x^2 + y^2 \le r^2\}$ is rotated about the line $x = R$. (This solid is called a *torus*.)

56. Calculate the volume of the solid obtained when the triangle with vertices $(2, 5)$, $(6, 1)$, and $(4, 4)$ is rotated about the line $x = -3$.

57. Calculate the volume obtained when the region outside the square $\{(x, y) : |x| < 1, |y| < 1\}$ and inside the circle $\{(x, y) : x^2 + y^2 \le 4\}$ is rotated about the line $y = 7$.

58. A solid has the ellipse $x^2 + 4y^2 = 16$ as its base. The vertical slices *parallel* to the line $y = 2x$ are equilateral triangles. Find the volume.

59. The base of a solid $\mathcal{S}$ is the disk $x^2 + y^2 \le 25$. For each $k \in [-5, 5]$, the plane through the line $x = k$ and perpendicular to the xy-plane intersects $\mathcal{S}$ in a square. Find the volume of $\mathcal{S}$.

60. A solid has as its base the region bounded by the parabola $x - y^2 = -8$ and the left branch of the hyperbola $x^2 - y^2 - 4 = 0$. The vertical slices perpendicular to the x-axis are squares. Find the volume of the solid.

61. Old Boniface he took his cheer,
Then he drilled a hole in a solid sphere
Clear through the center straight and strong,
And the hole was just ten inches long.
Now tell us, when the end was gained,
What volume in the sphere remained.
Sounds like you've not been told enough,
But that's all you need; it's not too tough.

62. An open cylindrical beaker with circular base has height L and radius r. It is partially filled with a volume V of a fluid. Consider the parameters L, r, and V to be constant. The axis of symmetry of the beaker is along the positive y-axis, and one diameter of its base is along the x-axis. When the tank is revolved about the y-axis with angular speed ω, the surface of the fluid assumes a shape that is the paraboloid of revolution that results when the curve

$$y = h + \frac{\omega^2 x^2}{2g},\ 0 \le x \le r,$$

is revolved about the y-axis. This formula is valid for angular speeds at which the surface of the fluid has not yet touched the base or the mouth of the beaker. The number $h = h(\omega)$ is in the interval $[0, V/(\pi r^2)]$ and depends on ω. (When $\omega = 0$, $h = V/(\pi r^2)$. As ω increases, h decreases.)

a. Find a formula for $h(\omega)$.

b. At what value ω_S of ω does spilling begin, assuming that $h(\omega) > 0$ for $\omega < \omega_S$?

c. At what value ω_B of ω does the surface touch the bottom of the beaker, assuming that spilling does not occur for $\omega < \omega_B$?

d. As ω increases, does the surface of the fluid touch the bottom of the beaker or the mouth of the beaker first?

Calculator/Computer Exercises

63. The region between the graphs of $y = x\exp(x)$ and $y = \sqrt{x}$ is rotated about the y-axis. Use Simpson's Rule to calculate the resulting volume to four decimal places.

64. The region below the graph of $y = \exp(-x^2)$, $-1 \le x \le 1$, is rotated about the x-axis. Use Simpson's Rule to calculate the resulting volume to four decimal places.

65. A flashlight reflector is made of an aluminum alloy that has mass density 3.743 g/cm^3. The reflector occupies the solid region obtained when the region bounded by $y = 2.530\sqrt{x}$, $y = 2.530\sqrt{x} + 0.300$, $x = 0$, and $x = 2.5$ cm is rotated about the x-axis. Sketch the cross section of the solid in the xy-plane. What is its mass?

66. The equation of the Gateway Arch in St. Louis, Missouri, is

$$y = 693.8597 - 34.38365(\exp(kx) + \exp(-kx))$$

for $k = 0.0100333$ and $-299.2239 < x < 299.2239$ where both x and y are measured in feet. Rotate this curve about its vertical axis of symmetry and compute the resulting volume.

8.2 Arc Length and Surface Area

Just as we have approximated area by a sum of areas of rectangles, so can we approximate the length of a curve by a sum of lengths of line segments. This process leads to an integral that is used to calculate the length of a graph of a function. We can apply analogous ideas to calculate the area of a surface of revolution.