Title: Population Growth

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Problem Statement: It is common to assume that a population will grow at a rate that is proportional to its size; that is, the larger the population, the larger its growth rate. If we denote population size by p and time by t we have a population growth model $\frac{dp}{dt} = kp$ (model 1). Solutions to this differential equation are of the form $p(t) = p_o e^{kt}$, where p_o is the initial ($t = 0$) population. The constant k is assumed to be a characteristic of the population under consideration, higher for rabbits than for people, for example. Used in the context of human population growth, model 1 is called the Malthusian Law, in honor of Thomas Malthus (1766-1834), author of an influential essay on overpopulation.

A second model of population growth was published by Raymond Pearl and Lowell Reed. They contended that it was absurd to try to predict population with any equation whose value continues to increase without bound; that the population of any confined geographic area must have an upper bound, say M, beyond which the population will not grow. Specifically, they hypothesized certain conditions for a satisfactory model:

1. Asymptotic to a line $p(t) = M$ when $t \to +\infty$.
2. Asymptotic to a line $p(t) = 0$ when $t \to -\infty$.
3. A point of inflection at some point $t = \alpha$ and $p = \beta$.
4. Concave upward to the left of $t = \alpha$ and concave downward to the right of $t = \alpha$.
5. Slope is never zero for any value of t.
6. Values of $p(t)$ varying continuously from 0 to M as t varies from $-\infty$ to $+\infty$.

A mathematical model which fits these conditions can be derived along the lines of model 1. Assume that there is an upper bound M beyond which the population will never grow, and that the rate of growth of $p(t)$ is proportional to the product of the population p and the difference $\frac{M-p}{M}$. (Note that this term will have very little influence for small values of p as it will be near 1 but will have a dampening effect for p near M as it will be near 0.) Model 2 can thus be written as $\frac{dp}{dt} = kp\left[\frac{M-p}{M}\right]$. Note that when the population is small, the population growth rate is close to that of model 1, but as the population size gets close to M, its growth rate becomes very small. Solutions to this differential equation are of the form $p(t) = \frac{Mp_o}{p_o + (M - p_o)e^{-kt}}$. Pearl and Reed used the equivalent equation $p(t) = \frac{be^{at}}{1 + ce^{at}}$ in their model.

Below is a set of questions concerning each model. Your task is to answer these questions and submit a written report of your findings.

(A) Model 1:

(1) Verify that the equation for $p(t)$ in model 1 does satisfy the differential equation for model 1.

(2) Use the data in Table 1 below to find a good choice for the constant k in model 1. Be careful to explain how you used the data in the table, what assumptions you made, and describe any limitations of your equation for $p(t)$ under model 1.

(3) Make a new column in table 1 listing predicted population under model 1. Describe how well your version of model 1 fits the population data from table 1.

(4) According to model 1, what will be the population of this country in 2000? in 2010?

(5) How well does model 1 work and why?

(B) Model 2:

(6) Verify that the equation for $p(t)$ does satisfy the differential equation.

(7) Verify that the two forms of the equation for $p(t)$ are equivalent.

(8) Pearl and Reed, beginning at 1780 and using the data for 1790, 1850 and a different figure for 1910 (91,972,266), published a population equation, $p(t) = \dfrac{2930300.9}{0.014854 + e^{-0.0313395t}}$. See if you can obtain the same equation using the same data points. Then check to see how well this function fits the data.

(9) Pearl and Reed came to the conclusion that the maximum population for our country is 197,274,000. How do you think they got this value? Is it correct?

(10) Pearl and Reed identified 1914 as the year in which the U.S. population curve would turn from concave up to concave down. How do you think they reached this conclusion?

(11) Using the data that has accumulated since Pearl and Reed did their work, try your hand at finding the parameter values to best fit model 2 to the data. State your resulting population equation, your predicted maximum population and the year in which the concavity of the population curve changes.

Table 1: U.S. Population (taken from *The World Almanac*, 1992)

year	population	year	population
1790	3,929,214	1890	62,979,766
1800	5,308,483	1900	76,212,168
1810	7,239,881	1910	92,228,496
1820	9,638,453	1920	106,021,537
1830	12,860,702	1930	123,202,624
1840	17,063,353	1940	132,164,569
1850	23,191,876	1950	151,325,798
1860	31,443,321	1960	179,323,175
1870	38,558,371	1970	203,302,031
1880	50,189,209	1980	226,542,203
		1990	248,709,873
		2000	281,421,906
		2010	308,745,538