# Lab 25: Differential Equations and Euler's Method

## Goals

- To illustrate differential equations as a modeling tool and as a major application of calculus.
- To solve differential equations numerically by Euler's method.
- To investigate how the accuracy of approximate solutions depends on the step size.

## Before the Lab

One of the major applications of calculus involves the formulation and solution of differential equations that arise in many fields of study. An example from physics, well supported by experimental data, is a model for radioactive decay. This model says that if $A(t)$ denotes the level of radioactivity of a substance at time $t$, then the rate of change in $A(t)$ is proportional to $A(t)$ itself. Let $r$ denote the proportionality constant. The value of this decay constant will depend on the particular radioactive material. Because we are modeling decay, $dA/dt$ will be negative. By requiring $r > 0$, we get the differential equation

$$\frac{dA}{dt} = -rA(t).$$

If we know the value of $A$ at $t = 0$, often written $A(0) = A_0$, we have what is known as an *initial-value problem.* General theorems from the theory of differential equations assure us that virtually any reasonable initial-value problem will have a solution and that the solution will be unique. This will certainly be true for the differential equations we encounter in this lab.

1. a. Think about a family of functions whose derivatives are just themselves times a multiplicative constant. Now guess a solution $A$ for the above differential equation that models radioactivity decay.

   b. Check that any constant multiple of a solution to the above differential equation is also a solution. Use the initial condition $A(0) = A_0$ to determine the constant multiple. Put your results together to record the unique function $A$ that satisfies the initial-value problem.

c. Try out your solution to the differential equation with decay constant $r = 2$ and initial value $A(0) = 10$. Compute $A(.1)$, $A(.2)$ and $A(1)$ as predictions of the levels of radioactivity at times $t = .1$, $.2$, and $1$.

As is the case with indefinite integrals, many differential equations have solutions that cannot be expressed in terms of familiar functions. Fortunately, solutions to differential equations can be approximated to almost any desired degree of accuracy by a variety of interesting and widely used techniques. Such an approximation is called a *numerical solution* to the differential equation. In this lab we will investigate a very simple and natural approximation technique, known as Euler's method, for producing a numerical solution. Even though more sophisticated methods are used in practice, Euler's method serves as an excellent introduction to the numerical solution of differential equations.

Let us consider the initial-value problem given by

$$\frac{dy}{dt} = f(t, y(t)), \qquad y(t_0) = y_0.$$

Here $y$ is an unknown function of $t$ that we seek as the solution to the differential equation, and $f$ is a known function that depends on both $t$ and $y$. The function $y$ is a solution in the sense that the derivative of $y$ evaluated at $t$ is precisely $f(t, y(t))$, the value we obtain from the function on the right hand side of the differential equations when we plug in $t$ and $y(t)$. Of course the solution $y$ must have the value $y_0$ at $t_0$, thus also satisfying the initial condition.

Euler's method gives an approximate solution for $t$ values near $t_0$ by using $\frac{y(t+h) - y(t)}{h}$ as an estimate for $y'(t)$. The increment $h$ is called the *step size*. Values of $h$ close to zero generally give better estimates for $y'(t)$ and hence better approximations.

2. a. Estimate $y'(t_0)$ by the difference quotient based on the increment in $t$ from $t_0$ to $t_1 = t_0 + h$. Convince yourself that this approximation converts the differential equation into the approximate equality $\frac{y(t_1) - y(t_0)}{h} \approx f(t_0, y_0)$.

   b. Use the approximation in part a to show that, at $t_1$, the exact value of the solution $y(t_1)$ can be approximated by $y_1 = y_0 + hf(t_0, y_0)$.

   c. Now let $t_2 = t_1 + h$ and $t_3 = t_2 + h$ so that in general $t_k = t_{k-1} + h = t_0 + kh$. Repeat the argument above to obtain an approximation to $y(t_2)$. In addition to approximating $y'(t_1)$ by the difference quotient, you will need to approximate $f(t_1, y(t_1))$ by $f(t_1, y_1)$. This is reasonable provided $f$ is continuous and $y_1$ is close to $y(t_1)$.



   d. Carry out the approximation argument one or more steps until you are convinced that the above process can be continued to approximate $y(t_{n+1})$ by

$$y_{n+1} = y_n + hf(t_n, y_n), \qquad n = 0, 1, 2, \ldots .$$

   This is the defining equation of Euler's method.

   e. What is the function $f$ in the differential equation for radioactive decay? Do not be alarmed that $t$ does not appear in the formula for $f$. Keep in mind that $r$ is a constant. If there is any chance you will interpret the decay constant as a variable, set it equal to 2 and work with that.

   f. Use parts b and c above with step size $h = .1$, decay constant $r = 2$, and initial value $A_0 = 10$ to estimate the solution to the radioactive decay equation at the points $t_1 = .1$ and $t_2 = .2$. Call these values $A_1$ and $A_2$. How do they compare with the values $A(t_1)$ and $A(t_2)$ of the true solution that you obtained in Problem 1.

## In the Lab

3. Your instructor will provide you with a program for applying Euler's method to obtain an approximate solution of a differential equation.

   a. Test this on the differential equation for radioactive decay with step size $h = .1$, decay constant $r = 2$, and initial value $A_0 = 10$. Do the approximations at $t = .1$ and $t = .2$ agree with your answers in Problem 2f?

   b. Use the program, again with $h = .1$, to estimate the solution at $t = 1$. Compare this estimate with the value of the true solution $A(1)$ obtained in Problem 1c. See if you can get your computer to compare the results of Euler's method at each step along the interval $[0, 1]$ with the true solution. Try to get both a numerical and a graphical comparison.

4. Let us consider the influence of the step size when Euler's method is applied to the differential equation $\frac{dA}{dt} = -rA(t)$. As before, take $r = 2$ and $A_0 = 10$. Try a variety of step sizes, some less than .1 and some greater than .1 to study the relation between the accuracy and the step size. Collect and organize enough data to make a convincing argument for your conclusions. What price do you pay for increased accuracy?

5. Let us now try Euler's method in a slightly more complicated situation. Suppose the rate of growth of an algae population depends on the number of algae cells and the temperature. Specifically, suppose a biologist determines that the population $P(t)$ at time $t$ satisfies the initial-value problem

$$\frac{dP}{dt} = .003P(t)(2 + 5\sin(\tfrac{\pi}{12}t)), \qquad P(0) = 45000.$$

The factor $2 + 5\sin(\frac{\pi}{12}t)$ has a period of 24 to model the daily temperature fluctuation when $t$ is measured in hours.

a. Use Euler's method to obtain a numerical approximation to the solution over a time interval of four days. Experiment to determine a reasonable value of the step size.

b. Present your results numerically and graphically.

c. Describe the qualitative features of your results that might be of interest to a biologist.

6. Listed below are three differential equations with initial conditions. One has an easily guessed solution, one has the solution $y(t) = t\cos t$, and one has no solution in terms of known functions.

i. $\frac{dy}{dt} = \cos t - y\tan t, \qquad y(0) = 0$

ii. $\frac{dy}{dt} = t\cos(yt), \qquad y(0) = 0$

iii. $\frac{dy}{dt} = t^2, \qquad y(0) = 0$

a. Solve the equation which has the easily guessed solution by making a good guess and showing that it satisfies the differential equation.

b. For the one with solution $y(t) = t\cos t$, show that $y(t)$ is indeed a solution. Also solve it numerically on an appropriate interval and compare your results.

c. For the remaining differential equation, solve it numerically on an interval of your own choosing.



## Further Exploration

7. What do you think happens if the step size in Euler's method is negative? Make a conjecture based on the geometric ideas behind the formula for Euler's method. Test your conjecture on the radioactive decay equation and other initial value problems from this lab.

8. As mentioned earlier, Euler's method is just a naïve beginning of the numerical fun one can have in solving differential equations. Two of the many numerical methods for solving initial-value problems are called the Improved Euler method and the Runge-Kutta method. Look up one of these methods in a textbook on differential equations or numerical analysis.

a. Give an intuitive explanation of how the method works.

b. Write or acquire a program to implement it.

c. Test it out on some of the equations in this lab.

d. Compare it to Euler's method for speed and accuracy.