Lab 24: Polar Equations

Goals

- To match elements of the polar graph $r = r(\theta)$ with corresponding values of θ.
- To develop mathematical formulas for slope and angle in polar coordinates, and to use them to address questions that lie beyond the reach of strictly graphical analysis.
- To match aspects of the graph $y = f(x)$ in rectangular coordinates against aspects of the graph $r = f(\theta)$ in polar coordinates.

Before the Lab

The purpose of this lab is to give you some experience in working with the graphs of equations defined by polar functions. We will use the fact that any polar function can be written in parametric form. If the polar coordinates of a point P are (r, θ), then the rectangular coordinates of P will be $(x, y) = (r\cos\theta, r\sin\theta)$. This leads to the standard parametrization of a polar curve $r = r(\theta)$ by

$$x(\theta) = r(\theta)\cos\theta,$$

and

$$y(\theta) = r(\theta)\sin\theta.$$

1. Consider the graph of a function defined parametrically by $x = x(t)$ and $y = y(t)$. The slope of curve at point $(x(t), y(t))$ is given by $\frac{y'(t)}{x'(t)}$. Use this result and the standard parametrization of a polar curve $r = r(\theta)$ given above to show that the slope of a polar graph at the point $(r(\theta), \theta)$ is given by
$$\frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}.$$

In the lab, a polar curve will be given as the graph of the function $r(\theta)$ in polar coordinates. You will need to be able to match points on the curve with their corresponding values of θ in the interval $[a, b]$. For example, points in the first quadrant will correspond to values of θ in the interval $[0, \frac{\pi}{2}]$ when $r(\theta)$ is positive on this interval.

2. What condition on r will force points in the first quadrant to correspond to values of θ between π and $\frac{3\pi}{2}$? Explain.

3. Given a polar graph of a function r, how can you determine which values of θ correspond to points $(r(\theta), \theta)$ where the curve crosses the x-axis?

In the Lab

4. *Flowers.*

   a. Use your graphing utility to verify that the polar graph of $r(\theta) = \sin 3\theta$, for θ in the interval $[0, 2\pi]$ is shaped like a flower with three petals. Would a domain smaller than $[0, 2\pi]$ produce the same graph? Explain. What values of θ correspond to the petal in the first quadrant?

   b. Sketch the polar graph of $r(\theta) = \sin 2\theta$, θ in $[0, 2\pi]$. How many petals do you see? Which values of θ correspond to the petal in the first quadrant? Could you have gotten the same graph from a smaller domain for θ this time?

   c. Experiment with the polar graphs of $r(\theta) = \sin k\theta$ for other integer values of k. Make a conjecture about the connection between the integer k and the number of petals in the polar graph.

5. *Limaçons.* Polar graphs of the functions $r(\theta) = 1 + k\sin\theta$ are called *limaçons*, French for snails, possibly because of their vague resemblance to the shape of these animals for certain values of k. See what you think of the resemblance by sketching the polar graph of $r(\theta) = 1 + 2\sin\theta$, for θ in $[0, 2\pi]$ (These were probably named by the same kind of people who call the big dipper Ursa Major).

   a. Your sketch for $k = 2$ should show a smaller loop inside a larger loop. Which values of θ correspond to this smaller loop?

   b. Sketch the limaçons for $k = .75$ and $k = .25$ to see other possible shapes. One should be dented, the other egg-like. Sketch the limaçons for several additional values of k. Besides being looped, dented, or egg-like, did you find any other shapes occuring as limaçons?

6. *Spirals.* If $r(\theta)$ is positive and is strictly increasing as a function of θ, the polar graph of r will be shaped like a spiral. For the spirals considered here, we will be mainly interested in their behavior as they cross the positive x-axis, for positive values of θ.

   a. *Logarithmic Spirals.* Sketch the polar graph of the function $r(\theta) = \ln\theta$ for values of θ in the interval $[1, 8\pi]$. For which values of θ does this curve cross the positive x-axis? What are the values of x at these points?

b. *Arithmetic Spirals.* Find a function r whose polar graph is a spiral which meets the positive x-axis at precisely the integer values: $1, 2, 3, \ldots$. Begin your search by listing the values of θ for which the spiral will cross the positive x-axis, and then consider what the value of $r(\theta)$ will need to be at these points. When you think you've got the right function, sketch your graph for values of θ in $[0, 8\pi]$ to be sure.

c. *Exponential Spirals.* Find a function r whose polar graph meets the positive x-axis at precisely the powers of $2 : 1, 2, 4, 8, \ldots$. Proceed as you did with arithmetic spirals, listing the values of θ for which the spiral crosses the positive x-axis, and then considering what the value for $r(\theta)$ will need to be at these points. Sketch your graph for values of θ in $[0, 8\pi]$ to confirm that you have the right function.

In the next five parts of this problem, we consider the angle at which a spiral meets the x-axis.

d. Use the formula you derived in Problem 1 to show that the slope of the line tangent to a spiral given by the function r at the points $(r(\theta), \theta)$ where it crosses the x-axis is given by $\dfrac{r(\theta)}{r'(\theta)}$.

e. Your sketch of the logarithmic spiral probably looked perpendicular to the x-axis at most of the points where it crossed the x-axis, but looks can be deceiving. Use the formula that you just derived to help explain the truth.

f. The sketch of your arithmetic spiral probably appears to meet the positive x-axis in the same angle at each point of intersection. Does it really? Explain.

g. The sketch of your exponential spiral probably also always appears to intersect the positive x-axis at the same angle. This time looks do not deceive. Explain.

h. Find a function r which gives a spiral that always crosses the x-axis in a $45°$ angle.

Further Exploration

7. *Limaçons Revisited.*

a. Select two values of k, one corresponding to a looped limaçon and one to a non-looped one. Use these values to sketch the graphs of $y = 1 + k \sin x$ in rectangular coordinates. Note those points (if there are any) where the curves cross the x-axis. Is there anything about these rectangular graphs that would account for the shape of the associated limaçons?

b. In Problem 5b, you found the shapes that the graph of a limaçon could have. Explain how you know that you found all of the shapes and determine exactly which values of k correspond to each shape. Analyzing graphs will not be sufficient to answer this question. You will need to use your formula for the slope of the tangent line to find those places on the limaçon where the tangent is horizontal.