Lab 23: Radius of Convergence for Power Series

Goals

- To obtain graphical evidence for the interval of convergence of a power series.
- To estimate the radius of convergence visually for selected power series.
- To see the radius of convergence as a function of the center about which the Taylor series is expanded.

Before the Lab

A series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ is called a *power series centered at c*. Associated with any such series is its *interval of convergence*: for all values of x interior to this interval, the series converges; for all values of x outside this interval, the series diverges. For the purposes of this lab, we ignore the behavior of the series for values of x at the endpoints of this interval. The constant c lies at the center of this interval. The distance from c to either endpoint is called the *radius of convergence* of the series. Your textbook illustrates how to use the root test and the ratio test to determine this radius precisely for certain series. The purpose of this lab is to give you a graphical context for these analytical facts.

1. Use the techniques developed in your textbook to compute the radius of convergence for each of the series below:

   a. $\sum_{n=0}^{\infty} 2^n(x-2)^n$

   b. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1}$

   c. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

One of the most important features of power series is their ability to represent familiar functions over their intervals of convergence. If the function f has derivatives of all orders at some point c, then we can form the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ where $a_n = \frac{f^{(n)}(c)}{n!}$. This is called the *Taylor series for f centered at c*. For most functions f that you are familiar with (and for all but one of the functions in this lab), the Taylor series will converge to $f(x)$ for all values of x within the interval of convergence. The partial sums for this series are polynomials of degree n:

$$p_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n.$$

In most of the standard cases, for values of x within the interval of convergence, $p_n(x)$ will be a good approximation for $f(x)$ when n is large enough. For values of x outside the interval of convergence, $p_n(x)$ is usually a poor approximation for $f(x)$, no matter how large n is.

For example, the Taylor series for the familiar natural logarithm function centered at 1 is $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1}$. You computed the radius of convergence of this series in Problem 1b above. Figure 1 superimposes the graph of $y = p_{16}(x)$ onto the graph of $y = \ln x$.

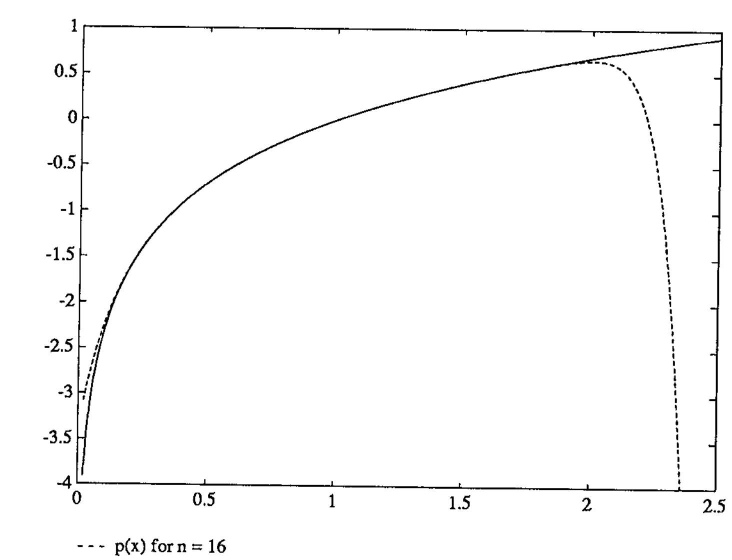

Figure 1: The natural logarithm function and its 16$^{\text{th}}$ Taylor approximation

Notice that $p_{16}(x)$ represents $\ln x$ pretty well over the interval of convergence, but that it pulls away sharply outside of that interval.

2. Highlight the interval on the x-axis in Figure 1 on which p_{16} seems to agree with the logarithm function.

The Taylor series for the arctangent function centered at 0 is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

You computed its radius of convergence in Problem 1c. The figure below superimposes the graph of $y = p_{15}(x)$ onto the graph of $y = \arctan x$.

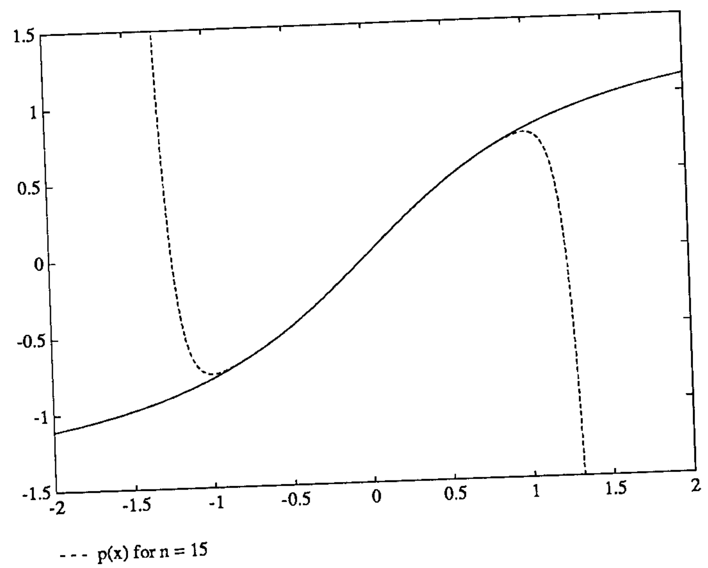

Figure 2: The arctangent function and its 15$^{\text{th}}$ Taylor approximation

3. Highlight the interval on the x-axis in Figure 2 on which p_{15} seems to agree with the arctangent function. How does this compare with the radius of convergence you computed in Problem 1c?

In the Lab

In the problems that follow, you will be asked to produce Taylor polynomials and to superimpose their graphs onto those of the functions they represent. We will consistently use p_n to denote the n^{th} degree Taylor polynomial for a function f centered at a point c. In each case, you will be asked to determine the radius of convergence for the associated Taylor series and to support your conclusion with graphical evidence similar to that above.

4. Let $f(x) = \ln x$ and $c = 2$.

   a. The Taylor series for f centered at $c = 2$ is $\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n2^n}$.
   Write the first four terms of this series to make sure you see the pattern.

   b. Use your computer to produce p_3 centered at 2 for $f(x) = \ln x$. Check that the coefficients agree with the ones you recorded for p_3 in part a. In the viewing rectangle $[0, 5] \times [-10, 2]$ plot both $y = \ln x$ and $y = p_3(x)$. Also have your computer generate p_6 and add its graph to the screen. Copy your graphs onto your data sheet. Be sure to label each curve and to indicate the units in the x and y directions. On the basis of this graphical evidence, highlight the interval of convergence in your sketch.

   c. Compute the radius of convergence for this series and discuss the extent to which it agrees with the graphical evidence.

5. Again let $f(x) = \ln x$. This time we will vary the value of c.

   a. Use your computer to find p_7 centered at $c = 5$. Plot it along with $y = \ln x$ in the rectangle $[0, 15] \times [-10, 5]$. Note the apparent interval of convergence. Plot p_n for several larger values of n to get a feeling for the behavior of these polynomials over the interval of convergence. When you have obtained a picture that illustrates your idea of the interval of convergence, copy it onto your data sheet, making sure to label each curve. Explain why this picture supports your conclusion.

   b. By now you have seen that the radius of convergence changes when the point c changes. Experiment with several other positive values of c. How does the radius of convergence seem to depend upon the value of c?

6. Let $f(x) = \sqrt[3]{1+x}$ and consider the Taylor series for f centered at $c = 0$, at $c = 1$, and at $c = -2$.

   a. Use the techniques of Problem 4 to estimate the radius of convergence for the Taylor series of f when centered at $c = 0$, at $c = 1$, and at $c = -2$.

   b. How does the radius of convergence seem to depend upon c for this function? What similarities between the graph of $y = \sqrt[3]{1+x}$ near $x = -1$ and the graph of $y = \ln x$ near $x = 0$ might account for the dependence you observe?

7. Let $f(x) = \arctan x$.

   a. The arctangent function and all of its derivatives are perfectly well-behaved for every real value of x. Nevertheless, the radius of convergence is finite. When $c = 0$, you have already seen that the radius is 1. Use the techniques established above to estimate the radii of convergence when $c = 1$ and when $c = 2$.

   b. Estimate the radii of convergence for other values of c between -2 and 2. Record your data as points on a graph showing the radius of convergence as a function of the location of the center.

Further Exploration

8. This problem introduces a function whose Taylor series converges everywhere, but not to the value of the function. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

   a. Use the computer to sketch a graph of this function on the interval $[-1, 1]$. For which values of x does $f(x)$ appear to be 0? For which values of x does $f(x)$ actually equal 0?

   b. It appears that $f'(0) = 0$. Sketch the graph of f'. What does $f''(0)$ appear to be? Use your computer to derive and then sketch some of the higher derivatives of f. What does $f^{(n)}(0)$ appear to be in each case?

   c. In this case, appearances are not deceiving. It is a remarkable fact that f has derivatives of all orders at 0, and $f^{(n)}(0) = 0$ for all n. Given this fact, what is the Taylor series for f centered at $c = 0$? What is the interval of convergence for this series? For which values of x does the series converge to $f(x)$?

9. Obtain a formula in terms of n for the coefficient of $(x - c)^n$ in the Taylor series for $f(x) = \ln x$. Use this formula to determine the radius of convergence for this Taylor series. How does the radius of convergence depend on c? How does this analytical result compare with the graphical results you found in Problem 4?

10. Although the derivatives of $f(x) = \arctan x$ are well-behaved for all real numbers, the same cannot be said for all complex numbers. For which complex numbers x is the function $f(x) = \dfrac{1}{1+x^2}$ undefined? How might this account for your results in Problem 6?