Lab 22: Approximating Functions by Polynomials

Goals

- To introduce the idea of one function being a good approximation to another.
- To prepare students for work on Taylor polynomials and Taylor series.

In the Lab

Polynomials can be easily evaluated at any point and their integrals are easy to find. This is not true of many other functions. Thus, it is useful to find polynominals that are good approximations to other functions.

In this lab we will find polynomials that approximate the exponential function. This function is important in mathematics and frequently appears in models of natural phenomena (population growth and radioactive decay, for instance). In these situations we need an easy way to approximate e^x for all values of x, not just for integers and simple fractions. Also, integrals involving the exponential function are important in statistics. For example, $\frac{1}{\sqrt{2\pi}}\int_0^{.5} e^{-x^2}\,dx$, which calculates the probability of a certain event that follows the "bell-shaped curve" of the normal distribution, simply cannot be evaluated in terms of the usual functions of calculus.

We will rely on the computer's ability to evaluate and graph the exponential function in order to determine polynomials that appear to be good approximations to this function. We will also use our polynomial approximations to compute integrals involving the exponential function.

1. We begin with a constant function that best approximates e^x near $x = 0$. Why is the graph of $y = 1$ the best constant approximation to the graph of $y = e^x$ near $x = 0$? That is, why would $y = 2$ or $y = -1$ be a worse approximation to $y = e^x$ near $x = 0$? Let us denote this polynomial approximation of degree zero by p_0.

2. Now we want to add a first degree term to p_0 to find a polynomial of the form $1 + ax$ that best approximates e^x near 0. Use your computer to graph $y = e^x$ and several candidates such as $y = 1 + .5x$, $y = 1 + .9x$, and $y = 1 + 1.2x$ on the same axes. Keep in mind that you are looking for the value of a so that $1 + ax$ best approximates e^x near 0. Thus you should favor a line that follows along the curve $y = e^x$ right at 0. You may need to change your scale to decide which line is better.

   In your notebook record which lines you tried and explain the criteria you used in choosing the line that gives the best approximation. Let $p_1(x) = 1 + ax$ denote your choice of the line that best approximates e^x near 0.

3. Next, find the second degree term bx^2 to add to p_1 to get a quadratic polynomial $p_2(x) = 1 + ax + bx^2$ that best approximates e^x near 0. Try to get a parabola that follows along the graph of $y = e^x$ as closely as possible on both sides of 0. Again, record the polynomials you tried and why you finally chose the one you did.

4. Finally, find a third degree term cx^3 to add to p_2 to get a cubic polynomial $p_3(x) = 1 + ax + bx^2 + cx^3$ that best approximates e^x near 0. This may not be so easy; you may have to change scale several times before you see why one polynomial is better than another.

5. Now that you have a polynomial that approximates e^x, try evaluating $p_3(.5)$ as a computationally simple way of estimating $e^{.5}$. How close is the polynomial approximation to the value of $e^{.5}$ as determined by a calculator or computer? Which is larger? How does the error at other points of the interval $[0, .5]$ compare with the error at $x = .5$? If you cannot distinguish between the graphs of $y = p_3(x)$ and $y = e^x$, you may want to plot the difference $y = e^x - p_3(x)$ with a greatly magnified scale on the y-axis.

6. Let us return to the problem of computing a definite integral such as $\int_0^{.5} e^{-x^2}\,dx$ for which the integrand does not have an antiderivative in terms of elementary functions. Since $p_3(x)$ approximates e^x, we can use $p_3(-x^2)$ to approximate e^{-x^2}.

   a. Evaluate $\int_0^{.5} p_3(-x^2)\,dx$ as an approximation to $\int_0^{.5} e^{-x^2}\,dx$.

   b. Use the numerical integration command on your computer to approximate $\int_0^{.5} e^{-x^2}\,dx$. How does this compare with your answer from part a?

7. An analytical method for approximating a function near a point leads to what are known as *Taylor polynomials.* The Taylor polynomial of degree n is determined by matching the values of the polynomial and its first n derivatives with those of the function at a particular point.

   a. Make a table to compare the values of p_3 and its first three derivatives with the values of e^x and its derivatives, all evaluated at $x = 0$. How close was your polynomial p_3 to being a Taylor polynomial?

   b. Determine the cubic Taylor polynomial for the exponential function. To do this, adjust the four coefficients so the values of the Taylor polynomial and its first three derivatives match those of e^x at $x = 0$. Plot this polynomial and your polynomial p_3. Compare how close they are to the graph of $y = e^x$ near 0.

Further Exploration

Suppose we want a polynomial that approximates a function over some **fixed** interval, rather than in some vaguely defined interval "near" a given point. This is important, for example, when we approximate a definite integral of a function by the integral of the polynomial over the interval. There are many different methods for approximating a function over a given interval. Therefore do not worry if your answers to Problem 8 are quite different from those obtained by other students. The point of this problem is for you to think about criteria you might use to judge the quality of an approximation. You may want to take a course in numerical analysis for further information about the surprising variety of techniques for approximating functions.

8. a. Determine the constant polynomial, the first degree polynomial, and the quadratic polynomial that you feel best approximate e^x on the whole interval $[-1, 1]$. Record your attempts. Do all three polynomials have the same constant term? Did you change the coefficient of x when you went from the straight line to the parabola? What criteria are you using to decide if one polynomial is a better approximation than another?

   b. Compare your criteria for deciding when a polynomial is a good approximation near a point, and your criteria for deciding when a polynomial is a good approximation over a given interval. How and why are your criteria different?