Lab 21: Limit Comparison Test

Goal

- To examine the Limit Comparison Test and its proper use.

In the Lab

A fundamental idea in the study of series is that of comparison. We look at $\sum_{n=1}^{\infty} \frac{n-2}{n^2}$ and feel that it should behave pretty much like $\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$, so we try to formalize this idea. We write $\frac{n-2}{n^2} < \frac{n}{n^2} = \frac{1}{n}$, and then we realize that this inequality goes the wrong way; $a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges. From this information, one is not allowed to conclude anything about $\sum_{n=1}^{\infty} a_n$.

Still, we have the feeling that we should be able to use what we know about $\sum_{n=1}^{\infty} \frac{1}{n}$. And our feeling is right! It's just that we need a variation on the comparison test. The Limit Comparison Test gives us what we need.

Limit Comparison Test: If $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number (greater than zero, but finite), then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

The goal of this lab is to give you a feel for how you choose a comparison series in order to use the Limit Comparison Test.

1. Use the Limit Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{n-2}{n^2}$ converges or diverges.

2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n}$.

   a. Use your computer to graph the functions $f(x) = \frac{1}{x^{3/2}+x}$ and $g(x) = \frac{1}{x^{3/2}}$ on the same axes. Set the scale on the horizontal axis so the x-values go from 1 to 100. We are interested in comparing the functions for large values of x, and when x is large, both $f(x)$ and $g(x)$ are quite small. Thus you should set a scale on the vertical axis to view the y-values between 0 and 1. Sketch the graphs in your notebook.

   b. Use your computer to plot the function f defined in part a, together with $h(x) = \frac{1}{x}$ on the same axes. Sketch the results in your notebook. You may have to experiment to find a range of y-values that will enable you to see both functions on the same axes.

   c. Use your computer to calculate $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ and $\lim_{x\to\infty} \frac{f(x)}{h(x)}$ for the functions in parts a and b above. Note the results.

   d. Look at the results of parts a through c and the statement of the Limit Comparison Test. Should you compare the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n}$ with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ or with $\sum_{n=1}^{\infty} \frac{1}{n}$? Explain your choice after considering which term dominates the denominator for large values of n.

   e. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n}$ converge? Use the Limit Comparison Test and your knowledge of p-series (series of the form $\sum \frac{1}{n^p}$) to justify your conclusion.

3. Let us consider a different series, $\sum_{n=1}^{\infty} \frac{1+n^2}{n^3+n^2}$. Experiment with graphs and limits until you find a p-series with the appropriate value of p so you can apply the Limit Comparison Test tc this new series. Use the same techniques that you did in answering Problem 2 above. Note your attempts and why you rejected the failures, as well as why you chose the p you did. Does this series converge or diverge? Why?

4. Look next at $\sum_{n=1}^{\infty} \frac{n+\sqrt[3]{n}}{n^{7/3}+n^2}$. Again experiment with graphs and limits in order to choose a suitable comparison series. This time consider the dominant term in the numerator as well as the dominant term in the denominator. If you ignore the other terms, what p-series does this series most resemble? Does the series converge or diverge?

5. All of the examples have been series of the form $\sum \frac{n^a+n^b}{n^c+n^d}$. To what series of the form $\sum \frac{1}{n^p}$ would you compare $\sum \frac{n^a+n^b}{n^c+n^d}$ in order to use the Limit Comparison Test? How do you decide on your answer?

Further Exploration

6. a. Adapt the ideas you developed in the lab to determine whether the series $\sum_{n=0}^{\infty} \frac{1}{3^n-2^n}$ converges. Find an appropriate geometric series and use the Limit Comparison Test

   b. Also determine whether $\sum_{n=0}^{\infty} \frac{2^n+5^n}{3^n+6^n}$ converges. Try to generalize your result to series of the form $\sum_{n=0}^{\infty} \frac{a^n+b^n}{c^n+d^n}$ for positive constants a, b, c, and d.

   c. What series would you use to test the convergence of $\sum_{n=0}^{\infty} \frac{1}{2^n+n}$? What is the result of the Limit Comparison Test for this series?

   d. How would you handle the series $\sum_{n=0}^{\infty} \frac{3^n+n^5}{n^3+5^n}$? Does this series converge?

7. Why does the Limit Comparison Test insist that $\lim_{n\to\infty} \frac{a_n}{b_n}$ be not only finite but positive? What can you conclude if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$?