Key: L(a, b, n) = left Riemann; R(a, b, n) = right Riemann; T(a, b, n) = trapezoid; S(a, b, n) = Simpson's

Type in your calculator: Define L(a, b, n) = (b - a)/n * Σ(y1(a + (k - 1)*(b-a)/n), k, 1, n)

you should see: Define l(a, b, n) = $\frac{b-a}{n} \cdot \sum_{k=1}^{n} y1\left(a + \frac{(k-1)\cdot(b-a)}{n}\right)$

type: Define R(a, b, n) = (b - a)/n * Σ(y1(a + k*(b-a)/n), k, 1, n)

see: Define r(a, b, n) = $\frac{b-a}{n} \cdot \sum_{k=1}^{n} y1\left(a + \frac{k\cdot(b-a)}{n}\right)$

type: Define T(a, b, n) = (b - a)/(2*n) * (y1(a) + y1(b) + 2*Σ(y1(a + k*(b-a)/n), k, 1, n - 1))

see: Define t(a, b, n) = $\frac{b-a}{2\cdot n}\left(y1(a) + y1(b) + 2\cdot \sum_{k=1}^{n-1} y1\left(a + \frac{k\cdot(b-a)}{n}\right)\right)$

type: Define S(a, b, n) = (b - a)/(3*n) * (y1(a) + y1(b) + 4*Σ(y1(a + (2*k - 1)*(b - a)/n), k, 1, n/2) + 2*Σ(y1(a + 2*k*(b - a)/n), k, 1, (n/2) - 1))

see: Define s(a, b, n) = $\frac{b-a}{3\cdot n}\left(y1(a) + y1(b) + 4\cdot \sum_{k=1}^{\frac{n}{2}} y1\left(a + \frac{(2\cdot k-1)\cdot(b-a)}{n}\right) + 2\cdot \sum_{k=1}^{\frac{n}{2}-1} y1\left(a + \frac{2\cdot k\cdot(b-a)}{n}\right)\right)$

To use these: enter the function you would like to compute with into y1 (it **must** be y1) on your y= screen. Then go to the home screen and type S(a, b, n), for example.

Lab 18: Numerical Integration

Goals

- To understand the geometry behind two methods of numerical integration, the Trapezoid Rule and Simpson's Rule.
- To gain a feel for the relative speeds of convergence of Riemann sums, Trapezoid Rule, and Simpson's Rule.

Before the Lab

Read through this lab and answer Problems 1, 2*a*, 3*abc*, and 4.

In the Lab

Suppose we want to calculate $\int_a^b f(x)\,dx$. The Fundamental Theorem of Calculus states that $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f. For example, $\int_0^{\pi/2} \cos x\,dx = \sin(\pi/2) - \sin(0) = 1$. However, integrals such as $\int_0^{\pi/2} \cos\sqrt{x}\,dx$ and $\int_0^1 \sqrt{1+9x^4}\,dx$ still give us problems since we cannot find usable expressions for the antiderivatives of the integrands. In this lab we will explore several methods for computing numerical approximations to such integrals.

Riemann Sums

One approach that you have seen in the definition of an integral is to form a Riemann sum. In this method, we replace the area under the curve $y = f(x)$, $a \leq x \leq b$, by the area of some rectangles. In Figure 1 we have a picture of a Riemann sum using four subintervals of equal length, with the height of each rectangle being the value of the function at the left-hand endpoint of that subinterval.

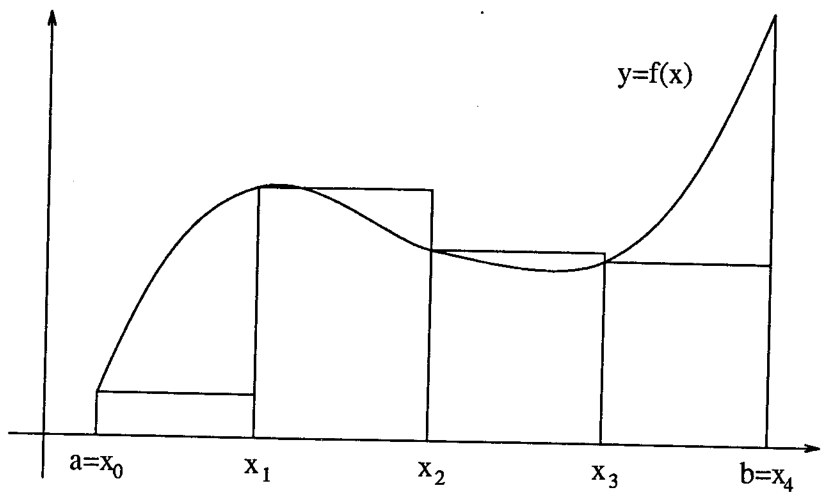

Figure 1: Riemann Sum

1. In this lab, we will be comparing several numerical answers for the value of $\int_0^1 5x^4 - 3x^2 + 1\,dx$ with the exact answer obtained by direct integration. What is the exact answer to $\int_0^1 5x^4 - 3x^2 + 1\,dx$?

2. Make a table to hold your answers for Problems 2, 4, and 5. Your table should include the method used, the number of subintervals, the approximation to the integral, the error in the approximation, and the width of a subinterval.

   a. Use a Riemann sum with $n = 4$ subintervals (by hand or calculator, showing all work) to approximate $\int_0^1 5x^4 - 3x^2 + 1\,dx$. Be sure to specify whether you followed a left-hand or right-hand rule.

   b. Use a Riemann sum program on your computer to approximate the same integral with $n = 16$ subintervals.

   c. Find a value for n so that the Riemann sum gives an answer that is accurate to 0.001.

Trapezoid Rule

In Riemann Sums, we replace the area under a curve by the area of rectangles. However, the corners of the rectangles tend to stick out. Another method is to form trapezoids instead of rectangles.

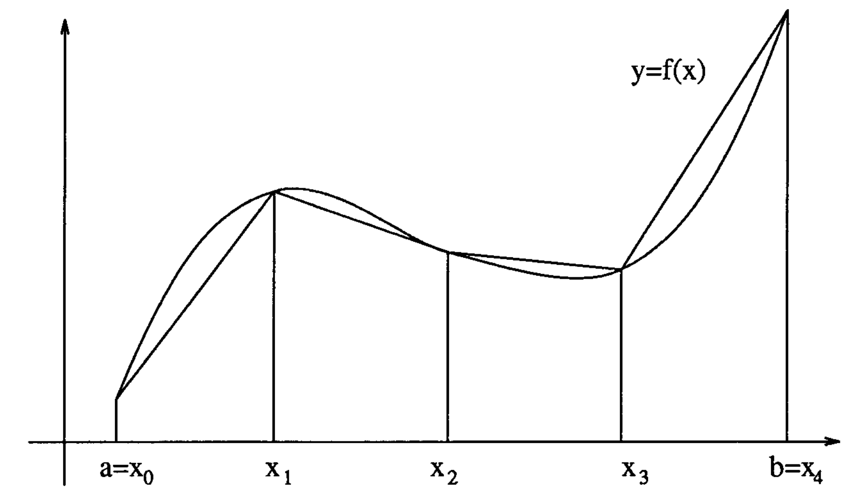

Figure 2: Trapezoid Rule

We will now develop the formula for the sum of the area of these trapezoids. This formula is known as the Trapezoid Rule.

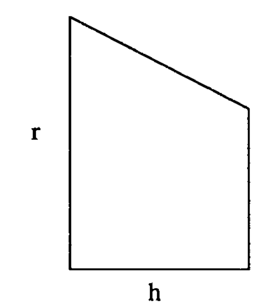

Figure 3

3. a. The formula for the area of a trapezoid is easy to derive. Divide the trapezoid in Figure 3 into a rectangle and a right triangle. The area of the rectangle is __________. The area of the triangle is __________. Show the algebra necessary to get the total area to be $\frac{h}{2}(r+s)$.

   b. Apply this formula four times to the four trapezoids in Figure 2. Let T_4 denote the sum of the areas of the four trapezoids. Show the algebra necessary to get that

$$T_4 = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\Big],$$

where $\Delta x = \dfrac{b-a}{4} = \dfrac{x_4 - x_0}{4}$.

c. If we use n equally spaced subintervals instead of 4, we let T_n be the sum of the areas of the n trapezoids. Derive a formula for T_n.

d. Repeat Problem 2 using the Trapezoid Rule, putting your data in the table.

Simpson's Rule

In the Trapezoid Rule, we replaced pieces of the curve by straight lines. In Simpson's Rule, we replace pieces of the curve by parabolas. To approximate $\int_a^b f(x)\,dx$, we divide $[a,b]$ into n equally spaced subintervals, where n is even. Simpson's Rule relies on the fact that there is a unique parabola through any three points on a curve. A picture of Simpson's Rule where $n = 4$ is given in Figure 4. The dashed line is the parabola through $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$, while the dotted line is the parabola through $(x_2, f(x_2))$, $(x_3, f(x_3))$ and $(x_4, f(x_4))$.

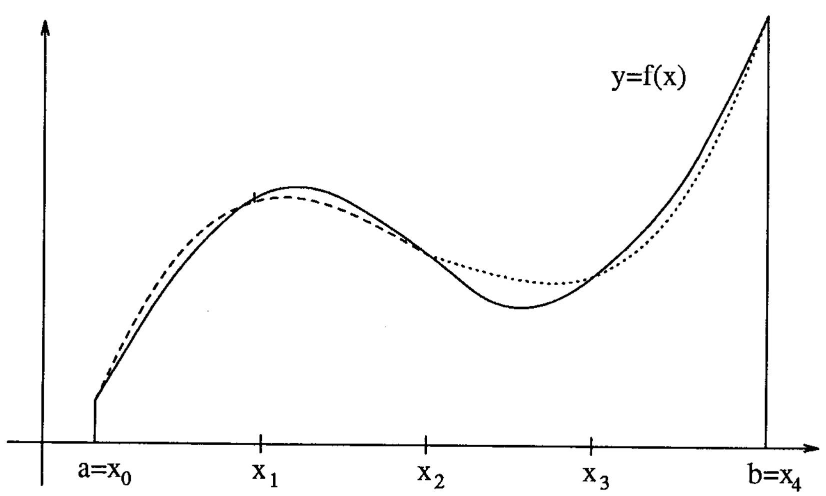

Figure 4: Simpson's Rule

The details are messy, but the area under the parabola through $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be shown to be $\dfrac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$, where $\Delta x = \dfrac{b-a}{4} = \dfrac{x_4 - x_0}{4}$.

4. a. Let S_4 be the sum of the areas under the two parabolas in Figure 4. Show the algebra necessary to get the formula for S_4:

$$S_4 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right].$$

   b. Let S_n be the sum of the areas under $n/2$ parabolas. Write a formula for S_n.

5. a. Repeat Problem 2 using Simpson's Rule, again using the table to organize your data.

   b. What value of n did you need for each method to get the answer to the desired accuracy? Which method needed the smallest value of n? (We call this the fastest method.) Which needed the largest? (This is the slowest method.)

   c. Recall that $\Delta x = \frac{b-a}{n}$. It is known that the error in these approximations is roughly proportional to $(\Delta x)^k$, where k is a positive integer. Using your table, find the value of k that you think works for the Riemann sum approximation. Repeat for the Trapezoid Rule and then for Simpson's Rule.

6. The previous problems have been artificial, since we were easily able to compute the integral exactly. As stated in the beginning, we often use numerical integration when we cannot apply the Fundamental Theorem of Calculus. Let us now investigate $\int_0^2 \sqrt{1+9x^4}\,dx$, an integral that we cannot do exactly. This integral arises in the calculation of the length of the curve $y = x^3$, where $0 \le x \le 2$.

   a. Approximate $\int_0^2 \sqrt{1+9x^4}\,dx$ using both the Trapezoid Rule and Simpson's Rule. Experiment with different values of n until you are convinced that your answers are accurate to 0.001.

   b. How did you decide when to stop? How do you get a feel for the accuracy of your answer if you do not have the exact answer to compare it to?

   c. Which method seems the fastest?

Functions given by data

Problem 6 illustrates the use of numerical integration to approximate $\int_a^b f(x)dx$ when it is difficult or impossible to find an antiderivative for f in terms of elementary functions. In applications it is often the case that functions are given by tables or by graphs, without any formulas attached. For these functions, we only know the function value at specified points. Numerical integration is ideally suited for integrating this type of function. Notice that in this situation we cannot possibly use the Fundamental Theorem of Calculus.

7. A map of an ocean front property is drawn in Figure 5. What is its area?

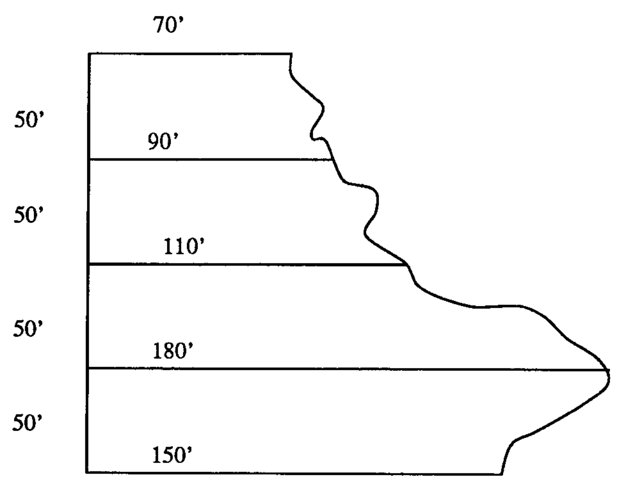

Figure 5

8. The data given below are adapted from the August, 1991 issue of *Road & Track* magazine. They give the velocity $v(t)$ of the \$239,000 Lamborghini Diablo at time t seconds.

t	$v(t)$
0	0 mph
1	14 mph
2	27 mph
3	40 mph
4	53 mph
5	64 mph
6	70 mph
7	77 mph
8	84 mph
9	90 mph
10	96 mph

Let $x(t)$ denote the distance the car travels at time t, $0 \le t \le 10$. Find $x(10)$. Discuss what method you used and how good you think your answer is.

Further Exploration

9. If $f''(x) > 0$ for all x in $[1, 4]$, would T_{12} be larger or smaller than the actual value of $\int_1^4 f(x)dx$? Why?

10. Although the Trapezoid Rule is usually an improvement over the left and right Riemann sums, it is related to them. Show that the Trapezoid Rule is the average of the left-hand and right-hand Riemann sums. You may do this either algebraically or geometrically.

11. If $f(x)$ is a linear function, it is easy to see that $T_1 = \int_a^b f(x)\,dx$. Similarly, if $f(x)$ is a quadratic, $S_2 = \int_a^b f(x)\,dx$. It is surprising, however, that $S_2 = \int_a^b f(x)\,dx$ if f is a cubic even though the approximating function is not an exact fit. Show that $S_2 = \int_0^1 x^3\,dx$.