Lab 15: A Mystery Function

Goals

- To investigate the effects of shrinking and stretching graphs.
- To use the idea of defining a function as the area under a curve.
- To develop some basic properties of a mystery function and use them to identify the function.

In the Lab

1. Recall how the period of the sine and cosine functions changes when the independent variable is scaled by a constant factor: if x is replaced by $\frac{x}{b}$, the graph is stretched horizontally by a factor of b changing the period from 2π to $2\pi b$. Go ahead and have the computer or calculator plot $y = \sin x$ and $y = \sin \frac{x}{3}$, for example, if you have any doubts about this.

   a. This stretching phenomenon does not depend on any special properties of the trigonometric functions or even on the fact that these functions are periodic. It actually works for any function. For example, use your computer or calculator to graph $g(x) = |x^3 - 6x^2 + 9x - 3|$ on the interval $[0, 4]$. Use the graph to approximate the x and y coordinates of all of the relative maxima and minima of this function on the interval $[0, 4]$. Now graph $y = g(\frac{x}{2})$ over the interval $[0, 8]$. What are the approximate x and y coordinates of all of the relative maxima and minima of this function on the interval $[0, 8]$? Repeat with the graph of $y = g(\frac{x}{.75})$ over the interval $[0, 3]$. Describe what happens to corresponding relative maximum and minimum points when x is replaced by $\frac{x}{b}$.

   b. Consider the regions bounded by the graph of $g(x) = |x^3 - 6x^2 + 9x - 3|$, the x-axis and the line $x = 4$. Use the graphs to determine how the total area of these regions compares with the total area of the regions bounded by the graph of $y = g(\frac{x}{2})$, the x-axis and the line $x = 8$. How do you think the total area of these regions compares with the total area of the regions bounded by the graph of $y = g(\frac{x}{.75})$, the x-axis and the line $x = 3$? Make a general statement about what you think happens to the area of any region when it is stretched horizontally by a factor of b.

   c. In a similar fashion, we can plot $y = \frac{1}{c}g(x)$ to shrink the graph of any function $y = g(x)$ vertically by a factor of c. Try this out on your computer with the function $g(x) = |x^3 - 6x^2 + 9x - 3|$ over the interval $[0, 4]$. Compare its graph with the graphs of $y = \frac{1}{2}g(x)$ and $y = \frac{1}{5}g(x)$ over the same interval. How do the x-coordinates and y-coordinates of the critical points of g compare with those of $\frac{1}{2}g$ and $\frac{1}{5}g$? How do the areas of the regions change as you change from g to $\frac{1}{c}g$?

We want to apply your observations about stretching and shrinking to the function $f(x) = \frac{1}{x}$. This function has the special property that when we stretch its graph horizontally by a factor of b and then shrink it vertically by the same factor, we return to the original graph. Indeed, $\frac{1}{b}f(\frac{x}{b}) = \frac{1}{b}(\frac{1}{\frac{x}{b}}) = \frac{1}{b}(\frac{b}{x}) = \frac{1}{x}$. We will define a mystery function M in terms of the area of a region whose upper boundary is the graph of $y = \frac{1}{x}$. Your job is to exploit the special property of this graph to prove some of the basic properties of this new function, thus identifying it as a function you have undoubtedly encountered in your precalculus course.

2. a. Define $M(x)$ to be the area of the region in the first quadrant under the graph of $f(x) = \frac{1}{x}$ and above the interval $[1, x]$. Use vertical lines through the endpoints of the interval to bound the region on the left and right. In your notebook sketch the regions whose areas correspond to $M(2)$, $M(3)$, and $M(6)$ to be sure you understand how the right endpoint of the interval $[1, x]$ serves as the independent variable in this new function. What is the domain of M? Using integral notation we can write $M(2) = \int_1^2 \frac{1}{x}\,dx$, $M(3) = \int_1^3 \frac{1}{x}\,dx$, etc.

   b. Your instructor will provide you with a method for approximating M. Use this method to estimate $M(x)$ for $x = .5$, 1, 1.5, 2, 2.5, 3, 4, 4.5, 7.5, 12. Feel free to trade some of these values with your neighbors in the lab. Record the results in a table. Can you explain why the minus sign appears in the value of $M(.5)$? Is your value of $M(1)$ reasonable in light of the geometric definition of M?

   c. Add your estimates of $M(1.5)$ and $M(2)$. What value on your table is your sum close to? Check for other sums that match other values. What relation is suggested by your observations? State this relationship as clearly as you can.

3. a. In your notebook sketch the graph of $f(x) = \frac{1}{x}$ and shade in the region whose area defines $M(a)$ for a typical value of a. Stretch the picture horizontally by a factor of b. As you discovered in Problem 1a, the resulting curve is the graph of $y = f(\frac{x}{b}) = \frac{b}{x}$. Make a second sketch to illustrate how the original shaded region stretches to the region below the new curve and above the interval $[b, ab]$ on the x-axis. How does the area of this region compare to the area of the original region?

   b. On a third sketch, shrink the graph of $f(\frac{x}{b})$ vertically by a factor of b. As you discovered in Problem 1c, the resulting curve is the graph of $\frac{1}{b}f(\frac{x}{b})$. As demonstrated in the paragraph before Problem 2, this simplifies to the original function $f(x) = \frac{1}{x}$. Shade in the region between the new curve and the interval $[b, ab]$ and explain why its area is equal to $M(a)$.

   c. On this third sketch, use a different color to shade in the region above the interval $[1, b]$ whose area represents $M(b)$. The union of the two shaded regions in this sketch has an area equal to the function M evaluated at what single number? State your conclusion as an identity involving $M(a)$, $M(b)$, and $M(ab)$.

4. Let us use the geometric definition of M to compute its derivative. Because we intend to establish a basic derivative formula for a new function, we will need to go back to the definition of the derivative as the limit of a difference quotient.

   a. Write the formula for $\frac{d}{dx}M(x)$ in terms of the definition of the derivative.

   b. Interpret the numerator as the difference between the areas of two regions, one contained within the other. Sketch a typical situation and shade in the vertical strip whose area represents the difference in the areas of these regions. Interpret the denominator as the width of the vertical strip. What is the geometric significance of the quotient?

   c. How does the limit of this quotient compare with the graph of $f(x) = \frac{1}{x}$? State your conclusion as a general formula for $\frac{d}{dx}M(x)$.

   d. Using the same method you used previously to approximate the function M, compute the difference quotients $\frac{M(2.1)-M(2)}{2.1-2}$ and $\frac{M(2.01)-M(2)}{2.01-2}$ as approximations to $M'(2)$. How do these values compare with the derivative formula you obtained in part c? Check some difference quotients that will approximate $M'(x)$ for some other values of x.

5. The function we are calling M is a function that you have seen before in your precalculus course. State what function you think M is by identifying a familiar function that satisfies the property established in Problem 3c or whose derivative satisfies the formula you discovered in Problem 4c. If you are stuck, use either a precalculus or calculus book to help you.

Further Exploration

6. Think of a as the product $\frac{a}{b} \cdot b$. Apply the identity you derived in Problem 3c to this product to derive an identity for $M(\frac{a}{b})$.

7. Another identity that M satisfies is $M(a^r) = r\,M(a)$. Of course, we need to assume $a > 0$ in order for M to be defined. Here is an outline for proving this relation.

   a. Check that the formula holds for the trivial cases $r = 0$ and $r = 1$. Use the multiplication formula for M to prove the new identity for $r = 2$, $r = 3$, and $r = 4$. Argue that once you have the formula for any whole number r, you can extend it to hold for the next whole number $r + 1$. Conclude that the formula holds for any positive integer r.

   b. Use your results so far to verify that $M(a^{-n}) = -n\,M(a)$ if n is a positive integer. Conclude that $M(a^r) = r\,M(a)$ for any integer r.

   c. If r is a rational number, we can write $r = \frac{p}{q}$ where p and q are integers. Write a as $a^{1/q}$ to the power q, and use the previous result to conclude that $M(a^{1/q}) = \frac{1}{q}M(a)$. Now continue the string of equalities $M(a^{p/q}) = M((a^p)^{\frac{1}{q}}) = \cdots$ ending with $\frac{p}{q}M(a)$. Be sure you can justify each step based on results from earlier steps in this lab.

   d. If you have a clear idea of what a^r means when r is irrational, use your definition of a^r to extend the identity to all real numbers r. Otherwise, don't worry if you have not seen a definition of a^r for irrational exponents. You are well on your way to bridging this gap.