Lab 14: Arc Length

Goals

- To develop the idea of approximating arc length by the sum of the lengths of straight line approximations to the curve.
- To compute arc length using an integral.

Before the Lab

The idea of the arc length of a curve is very easy to understand. You have had experience calculating the length of straight lines and circles. Intuitively, the length of the curve on the left in Figure 1 should be the length of the straight line we get on the right if we take the ends of the curve and pull it taut. Although this idea is easy to understand, it is not much good as a practical method of computation.

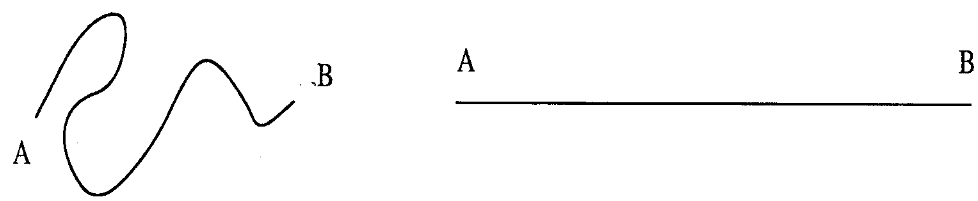

Figure 1

1. We start by reviewing the method for calculating the length of a straight line segment.

   a. Find the length of the straight line segment from $(1, 2)$ to $(5, 4)$, shown in Figure 2. The lengths of the two sides of the right triangle are $4 - 2 = 2$ and $5 - 1 = 4$. You can use the Pythagorean Theorem to determine the length of the hypotenuse.

   b. Now generalize this example to find a formula for the length of the segment of the straight line $y = mx + c$ from $x = a$ to $x = b$. This is the line segment connecting the points $(a, \_\_\_\_\_\_)$ and $(b, \_\_\_\_\_\_)$. Show the necessary work to get the length equal to $(b - a)\sqrt{1 + m^2}$. Check your answer to part a using this formula.

   c. Write an expression for the length of the curve in Figure 3 made up of four straight line segments with slopes m_1, m_2, m_3 and m_4.

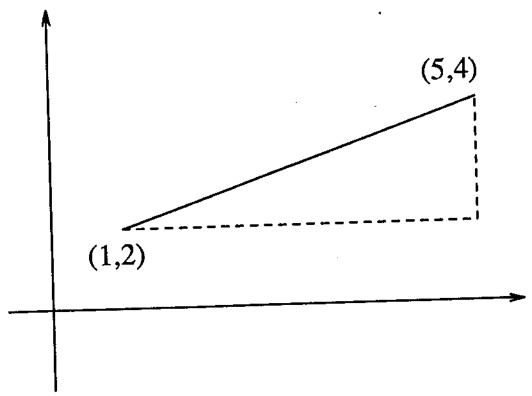

Figure 2

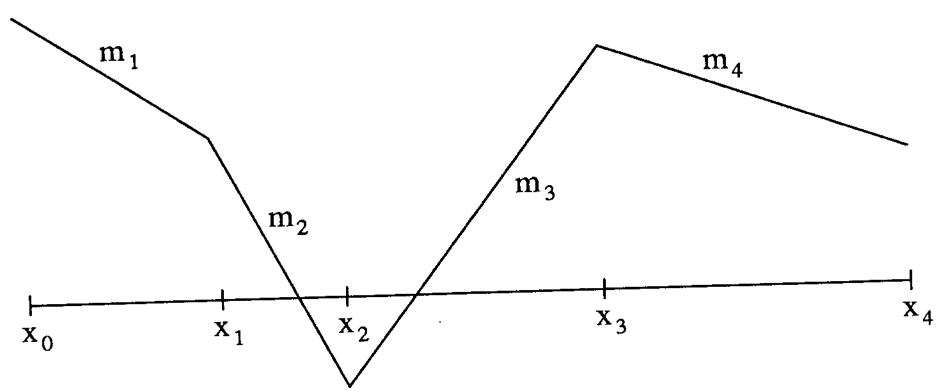

Figure 3

2. We are now ready to try to find the length of a curve defined by $y = f(x)$ for $a \le x \le b$, where the curve is not necessarily a straight line. Consider, for example, the graph $y = \sin x$ for $0 \le x \le \pi$, shown in Figure 4.

   a. A crude approximation to the length of the curve would be the straight line distance between the endpoints of the curve. Compute this distance. Is this value bigger or smaller than the actual arc length of this graph?

   b. A better approximation would be found by dividing $[0, \pi]$ into two pieces and adding the straight line distance from $(0, 0)$ to $(\frac{\pi}{2}, 1)$ to the distance from $(\frac{\pi}{2}, 1)$ to $(\pi, 0)$. See Figure 4. Compute this distance. How is it related to the answer in part a? How is it related to the actual arc length?

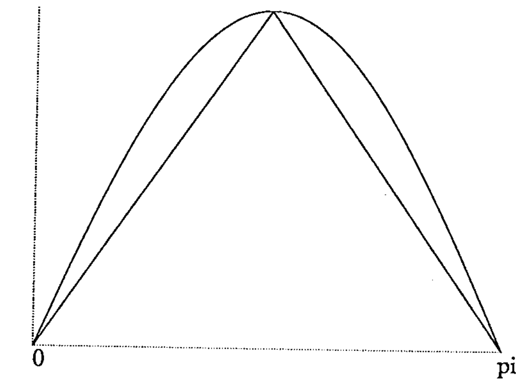

Figure 4

   c. To get a more accurate answer, we will subdivide $[0, \pi]$ more finely yet. Copy the graph of $y = \sin x$ from Figure 4 into your notebook and sketch the straight line approximation to the curve using four subintervals of equal size. On another copy of this curve, sketch the straight line approximation to the curve using eight subintervals of equal size. In the lab, you will use the computer to calculate the lengths of these approximations.

In the Lab

3. Use the computer commands provided by your instructor for calculating straight line approximations to the arc length. For each of the following four functions, determine the length of the straight line approximations to arc length using 2, 4, 8, 25, 50, and (if your computer is fast enough) 100 subintervals of equal size. Make your best guess as to the length of the curve, accurate to at least two decimal places. Record your data in a suitably labeled table.

   a. $f(x) = \sin x$ for $0 \le x \le \pi$

   b. $f(x) = \sqrt{9 - x^2}$ for $0 \le x \le 3$

   c. $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ for $1 \le x \le 3$

   d. $f(x) = e^x$ for $0 \le x \le 1$

   e. You can find the exact answer for the arc length of one of these curves without using calculus or the computer. Which curve is it, and what is the exact answer? (Hint: One is the equation of a well-known geometric object.)

It is possible to write an integral for the length of a curve $y = f(x)$ for $a \le x \le b$. This is done carefully in your textbook using the Mean Value Theorem. The derivation relies on the geometric ideas we have been using in this lab. We have seen in Problems 1 and 2 that the arc length is approximated by

$$\sum_{k=1}^{n} \sqrt{1+m_k^2}\left(\frac{b-a}{n}\right) = \sum_{k=1}^{n} \sqrt{1+\left(\frac{\Delta y_k}{\Delta x}\right)^2}\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and the slope of the k^{th} straight line segment is $\dfrac{\Delta y_k}{\Delta x}$. As $\Delta x \to 0$, we know that $\dfrac{\Delta y_k}{\Delta x} \to f'(x)$. Therefore the arc length equals

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} \sqrt{1+\left(\frac{\Delta y_k}{\Delta x}\right)^2}\,\Delta x = \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

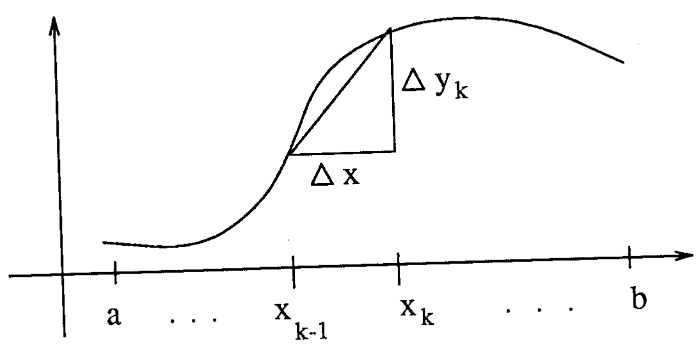

Figure 5

For example, $\int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$ gives the length of the curve $f(x) = \sin x$ for $0 \le x \le \pi$. Convince yourself that this is the correct integral.

Although we now have an integral formula for arc length, in practice it is difficult (or impossible) to apply the Fundamental Theorem of Calculus to most of the resulting integrals. This leaves us needing a numerical integration technique to approximate the value. For example, we cannot integrate $\int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$ exactly, but we can use a computer to find that the arc length of $y = \sin x$ for $0 \le x \le \pi$ is approximately 3.8202.

4. a. Write the appropriate integrals for the arc lengths for the curves given in Problems 3*b*, 3*c*, and 3*d*.

   b. The integral for Problem 3*c* can be computed by hand. Do it.

   c. Use your computer to approximate the value of the arc length integrals for the functions given in Problems 3*b* and 3*d*.

   d. Compare your answers to those you obtained in Problem 3.

Further Exploration

5. An approach to finding the arc length that people sometimes think of is the staircase method. This method is to partition $[a, b]$ into n pieces and approximate the arc length by the sum of the vertical and horizontal lines. See Figure 6. Unfortunately, this method does not work. Why not? What happens to the picture as n becomes arbitrarily large? What happens to the sum as n increases to infinity?

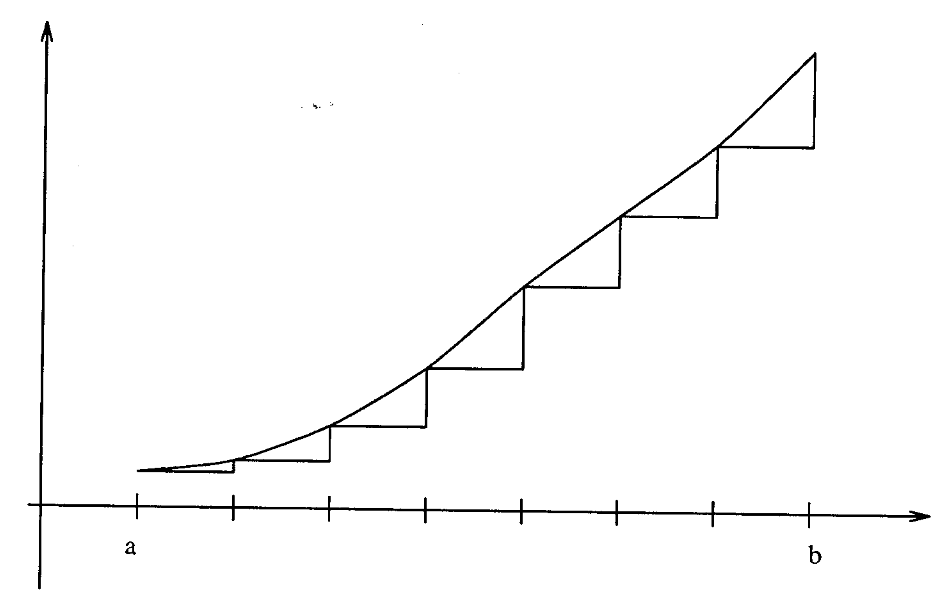

Figure 6