Lab 10: Indeterminate Limits and l'Hôpital's Rule

Goals

- To recognize limits of quotients that are indeterminate.
- To understand l'Hôpital's rule and its application.
- To appreciate why l'Hôpital's rule works.

Before the Lab

In this laboratory, we are interested in finding limits of quotients in cases which are referred to as *indeterminate*. This occurs, for example, when both the numerator and the denominator have limit 0 at the point in question. We refer to this kind of indeterminacy as the $\frac{0}{0}$ case.

One of the standard limit theorems allows us to compute, under favorable conditions, the limit of a quotient of functions as the quotient of the limits of the functions. More formally,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

provided limits for f and g exist at a and $\lim_{x\to a} g(x)\neq 0$.

1. For three of the limits below, use the result stated above to evaluate the limits without the help of a calculator or computer. For each of the other three, explain why the limit theorem does not apply, say what you can about the limit of the quotient, and indicate which are indeterminate in the $\frac{0}{0}$ sense described above.

   a. $\lim_{x\to 2}\dfrac{x^2+1}{x-4}$

   b. $\lim_{x\to 0}\dfrac{e^x-1}{\sin x+3}$

   c. $\lim_{x\to 2}\dfrac{\sqrt{x^2+5}}{\sin \pi x}$

   d. $\lim_{x\to 2}\dfrac{1-\frac{2}{x}}{\sin \pi x}$

   e. $\lim_{x\to 3}\dfrac{x^2-9}{\sqrt{1+x}}$

   f. $\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$, where f is any function that is differentiable at a

In the Lab

2. a. Plot the function $\dfrac{\ln x}{x^2-1}$ and, from the graph, determine or estimate the value of the indeterminate $\lim_{x\to 1}\dfrac{\ln x}{x^2-1}$.

   b. Repeat the procedure suggested in part a to obtain the limit for Problem 1*d*.

If indeterminate quotients were always of such a specific nature and we were able and willing to use graphical or numerical estimates, there would be little need to pursue these matters further. Often, however, indeterminates occur in more abstract and general situations. Thus we seek a correspondingly general approach that will apply to a wide variety of indeterminate situations.

3. The result we will be exploring is known as *l'Hôpital's Rule*. One version of it says that if $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ is indeterminate, but f and g have derivatives at a with $g'(a)\neq 0$, then $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{f'(a)}{g'(a)}$.

   a. By taking the appropriate derivatives, apply the above result to the quotient $\dfrac{\ln x}{x^2-1}$ at the point $a=1$. Also check the result graphically by plotting, on the same axes, the quotient of the functions f and g defined by $f(x)=\ln x$ and $g(x)=x^2-1$ along with the quotient of their derivatives (not the derivative of the quotient!) as called for by l'Hôpital's Rule. What should happen, according to l'Hôpital's Rule, in your graphs? Does it indeed happen?

   b. Do the same as above for the quotient of functions given in Problem 1*d*.

4. In this question we look at why the $f'(a)$ and $g'(a)$ arise in l'Hôpital's Rule. We restrict ourselves to the indeterminate case where both numerator and denominator approach 0 (*i.e.*, the $\frac{0}{0}$ case). Thus we assume that

   i. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, and

   ii. f and g are both differentiable at a with $g'(a) \neq 0$.

   a. Carefully explain why the assumptions made above ensure that $f(a) = 0$ and $g(a) = 0$.

   b. Under the above assumptions, carefully justify each of the equalities in the line below.

$$\frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}.$$

   c. Now build upon parts a and b to complete a proof for the $\frac{0}{0}$ case of l'Hôpital's Rule.

5. *Fun and games with l'Hôpital's Rule.*

   a. Use l'Hôpital's rule to compute $\lim_{x\to 0} \dfrac{1-e^{3x}}{\sin x + x}$. If you can, use the `limit` command on your computer to check your answer.

   b. Let f be a function that is differentiable at a. Perhaps the most famous indeterminate of all is $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$. What does l'Hôpital's rule say in this situation? Are you surprised?

   c. Make a conjecture about continuing the procedure called for in l'Hôpital's rule in situations where both $f'(a)$ and $g'(a)$ are also 0. Apply your conjecture to compute $\lim_{x\to 0} \dfrac{x-\sin x}{1-\cos x}$.

6. Consider the sector of a unit circle with angle x (in radians) as pictured in Figure 1. Let $f(x)$ be the area of the triangle ABC, while $g(x)$ is the area of the curved shape ABC.

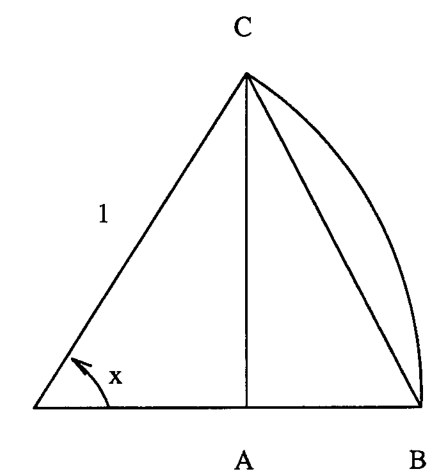

Figure 1

   a. By thinking geometrically, try to make a guess about the limit of $f(x)/g(x)$ as the angle x approaches 0.

   b. Show that $f(x) = \frac{1}{2}(\sin x - \sin x \cos x)$ and $g(x) = \frac{1}{2}(x - \sin x \cos x)$.

   c. Using the result stated in part b, compute the actual limit you guessed at in part a.

Further Exploration

7. In Problem 4 you developed an analytic proof of l'Hôpital's rule. In this problem, you will develop a geometric justification that resembles l'Hôpital's original argument.

   a. To start with an easier case, suppose f and g are straight lines that go through the point $(a, 0)$ with slopes m_1 and m_2, respectively. Therefore the equations for f and g are $f(x) = m_1(x-a)$ and $g(x) = m_2(x-a)$. Sketch a picture representing this situation. What is $\lim_{x\to a} \dfrac{f(x)}{g(x)}$? (Do not use l'Hôpital's rule to evaluate this limit!)

b. Now assume that f and g are functions that satisfy the same conditions as those listed in the statement of Problem 4. Figure 2 shows the graphs of two such functions. If we zoom in near the point $(a, 0)$ we will see two straight lines. What are the slopes of those lines? Use these results along with the results from part a to complete a geometric justification for l'Hôpital's rule.

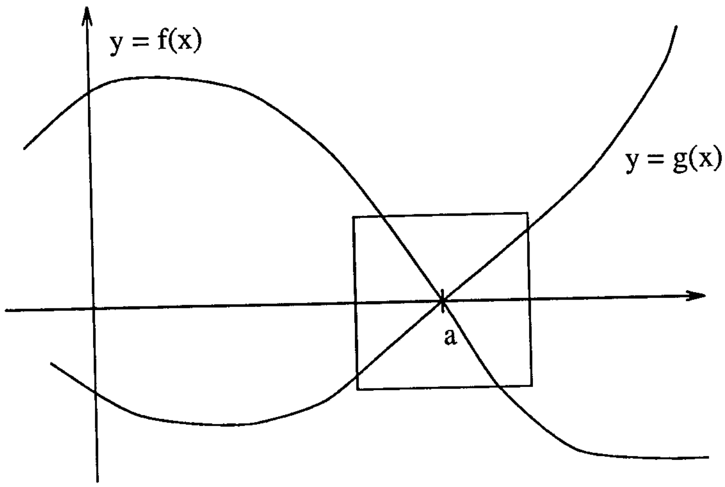

Figure 2

8. In checking l'Hôpital's rule graphically in Problem 3, you no doubt found that $\frac{f(x)}{g(x)}$ and $\frac{f'(x)}{g'(x)}$ approach the same value as x approaches a. This question asks you to investigate the *slopes* of these two quotient functions.

   a. By looking at the graphs in Problem 3 and other $\frac{0}{0}$ situations of your own choosing, try to discover a relationship between the above slopes.

   b. By using appropriate calculus techniques, prove your conjecture in part a. Hint: With $h(x) = \frac{f(x)}{g(x)}$ and $k(x) = \frac{f'(x)}{g'(x)}$, look carefully at and compare $h'(x)$ and $k'(x)$ as x approaches a.