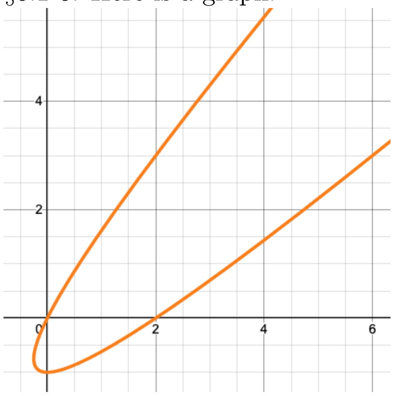
222 Assignment 7 Solutions

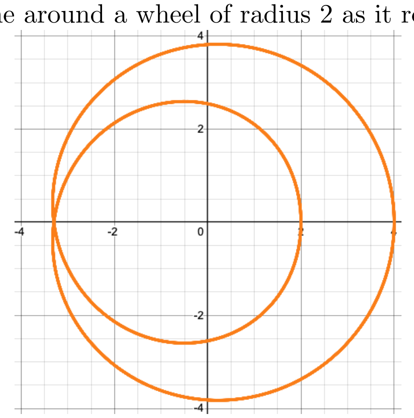
It's the last one. Hooray! And ... kinda sad (for me, at least). Many of these are just "make graphs" so you can see what our pretty pictures look like.

§6.1 6. Here is a graph:



What curve does it look like? Here's a much more interesting question .. how would you prove that it is (or isn't)? Moreso that any other question I've offered, I don't expect anyone to do this, but I will give +4 if anyone does.

§6.1 58. This question is not well written. There is no c in the original question. If the point is on the outside of the outer wheel, then c must be the same as b. If it isn't then c is the distance from the centre of the outer circle. The way the question is written with the values is the path of a point traveling a distance one around a wheel of radius 2 as it rolls around an inner circle of radius 1. Here is the picture:



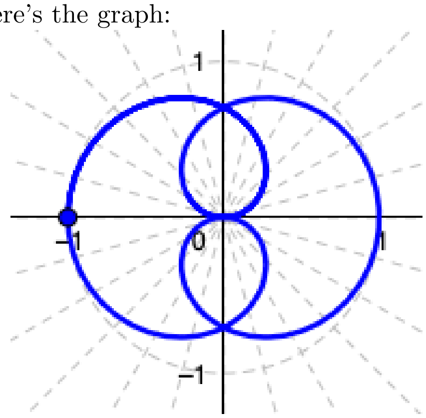
The inner circle is of radius 1. The centre of the outer wheel travels at radius 3, and then the point rolls at distance 1 around that centre.

§6.2 15. The curve is given as $(t\ln t, \sin^2 t)$. At $t = \frac{\pi}{4}$ the curve passes thru $(\frac{\pi}{4}\ln(\frac{\pi}{4}), \frac{1}{2})$. First we need the slope of the tangent line. It is $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sin t\cos t}{\ln t+1}$. At $t = \frac{\pi}{4}$ the slope is then: $\frac{2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{\ln\frac{\pi}{4}+1} = \frac{1}{\ln\frac{\pi}{4}+1}$. The tangent line is thus $y = \frac{1}{\ln\frac{\pi}{4}+1}(\frac{\pi}{4}\ln(\frac{\pi}{4}) - x) + \frac{1}{2}$, not much to see here.

I seemed to have omitted them. Please make sure that you can find area under parametric curves also, e.g. §6.2 41-46.

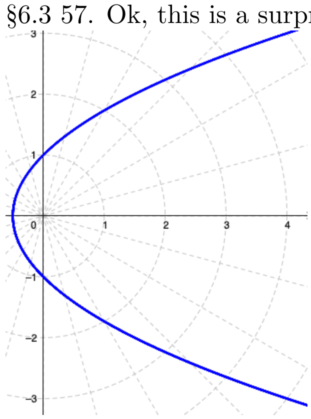
§6.2 49 Find arclength, ok. The formula is $\int_a^b \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}$. Our curve is $(\cos(2t), \sin(2t))$ for $0 \le t \le \frac{\pi}{2}$, which is an upper semicircle of radius one taken at double speed. It's kinda dull. What is the circumference? What is half? Ok, so this shouldn't be surprising, let's make sure we get the right answer: $\int_a^b \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2} = \int_0^{\pi/2} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}dt$. Notice that $\sin^2 a + \cos^2 a = 1$ for *any* a, so that works here and suddenly we have the integral of just ... 2. Ok, this is getting dull. So, the arclength is $\int_0^{\pi/2} 2dt = \pi$, as we thought. It's too bad #50 was impossible, as it would have been more interesting.

§6.3 51. Some more making graphs and pretty pictures. I recommend also making a graph of $y = 3\cos(x/2)$ and comparing them to see how and why the graph is what it is. Notice this graph requires a range of 4π to get the entire graph. It is symmetric over both axes and therefore also has half-turn symmetry. Here's the graph:



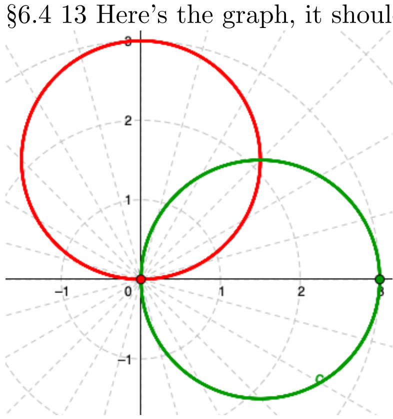
You should have been able to make this one by hand. Be sure that you practice doing so.

§6.3 57. Ok, this is a surprisingly familiar looking picture:



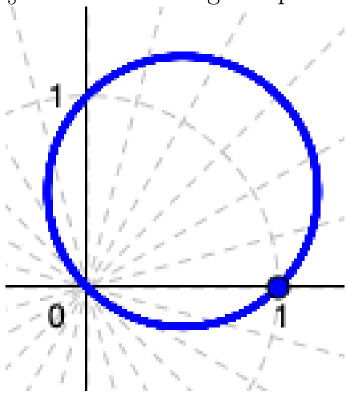
I will give +2 to the first person who can show me algebra work to convince me that this *is* what we think it is. It is much easier than the one above.

§6.4 13 Here's the graph, it should not be surprising by this point:



This is pretty simple. They intersect at $\theta = \pi/4$. There is also clearly symmetry over that line. It did ask for one integral, so I think the answer should be $\int_0^{\pi/4} 9\sin^2\theta d\theta$. Notice the cancelation of 2 for 2 copies and $\frac{1}{2}$ from the formula. Another answer could be $\frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} 9\cos^2\theta d\theta$. That's correct, but not as nice. And sure, it's not very interesting, but I'll give +1 to the first person who sends me the work to find the area.

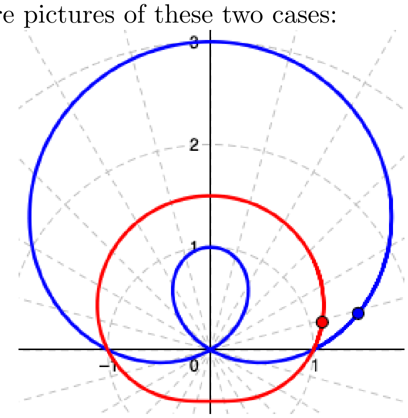
§6.4 45. Last assigned question from the textbook. First a graph of $r = \sin\theta + \cos\theta$:



It's a circle. Ok, so, if we knew the radius we could find the circumference. From the picture we see that $(1,0)$ and $(0,1)$ and $(0,0)$ are all on the circle. They are equidistant from $(\frac{1}{2},\frac{1}{2})$. That is the centre. The distance is $\frac{1}{\sqrt{2}}$. And that is the radius. So, the circumference is $2\pi\frac{1}{\sqrt{2}} = \pi\sqrt{2}$. That's the non-calculus way. Now, what about calculus? You say we didn't do arclength of polar curves? No, we didn't, but we did arclength of parametric curves, and this is just a special case. Since we will need it, what range of values gives the circle above? Both the functions repeat at π. We can also run tests from our graph to see it is so (I liked looking at the graph on [0,3]). The parametric formula is $\int_a^b \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}$. We will work in terms of θ, so $t = \theta$, and remember $x = r\cos\theta$ and $y = r\sin\theta$. and $r = \cos\theta + \sin\theta$. So, $x = \cos^2\theta + \cos\theta\sin\theta$ and $y = \sin^2\theta + \cos\theta\sin\theta$. So, we take derivatives, $\frac{dx}{d\theta} = -2\cos\theta\sin\theta + \cos^2\theta - \sin^2\theta$ and $\frac{dy}{d\theta} = 2\cos\theta\sin\theta + \cos^2\theta - \sin^2\theta$. Next square and add. And then ... a miracle occurs! Or something like that. Check the algebra carefully, but for our last celebration of the semester (at least together), all that algebra magically combines to equal ... 2. That's before taking the square root, so our integral becomes $\int_0^\pi \sqrt{2} = \pi\sqrt{2}$, like we said above. Ta da!

Doing this with the formula derived in class saves *some* work, but it's not immediate. $r(\theta) = \sin\theta + \cos\theta$, and $r'(theta) = \cos\theta - \sin\theta$. Using our class-formula $\int_0^\pi \sqrt{r(\theta)^2 + r'(\theta)^2}d\theta = \sqrt{(\sin\theta + \cos\theta)^2 + (\cos\theta - \sin\theta)^2}d\theta$. Because of all the trig, this will play out very similar to how it did in class. It squares to expand, but there is nice cancelation. The middle terms cancel and there is a $\sin^2\theta + \cos^2\theta$ from each of the squares. That produces $1 + 1 = 2$ and we end up back at $\int_0^\pi \sqrt{2} = \pi\sqrt{2}$.

Lab 24. 7. Since there was only one question to choose from, I felt it only fair that I write a solution for it. a. Ok, $r = 1 + 2\sin\theta$ has a loop and $r = 1 + \frac{1}{2}\sin\theta$ does not. Looking at the auxiliary graphs of $y = 1 + k\sin x$ the loops are formed where the auxiliary graph crosses the axis, i.e. when r is negative. Here are pictures of these two cases:



b. One other shape we get is a circle, when $k = 0$. Bigger k values make bigger loops. $k = 1$ is dented. $k = 0.75$ has a bend in, but not very sharply. Negative values aren't very interesting, they just reflect the graph over the x-axis. It looks as if $k = 0.5$ is the boundary between bending in and not, by looking at $k = 0.6$ and $k = 0.4$. But, let's do what they suggest since $r = 1 + k\sin\theta$, and $r' = k\cos\theta$ ... the slope at θ is, apparently $\frac{k\cos\theta\sin\theta+(1+k\sin\theta)\cos\theta}{k\cos^2\theta-(1+k\sin\theta)\sin\theta} = \frac{\cos\theta+2k\sin\theta\cos\theta}{k(\cos^2\theta-\sin^2\theta)-\sin\theta}$. Now, there's a couple identities hiding there. $2\sin\theta\cos\theta = \sin(2\theta)$ and $\cos^2\theta - \sin^2\theta = \cos(2\theta)$. So, putting both of those in, the slope is $\frac{\cos\theta+k\sin 2\theta}{k\cos 2\theta-\sin\theta}$. That's cute, but, not very helpful. Let's look back at the stage before that, especially at the numerator to find when it is zero: $\frac{\cos\theta+2k\sin\theta\cos\theta}{k(\cos^2\theta-\sin^2\theta)-\sin\theta}$. Let's set the numerator equal to zero $\cos\theta + 2k\sin\theta\cos\theta = 0$. We can factor to get $\cos\theta(1 + 2k\sin\theta) = 0$. We get values where $\cos\theta = 0$, that's at $\pi/2$ and $3\pi/2$, which don't surprise us. Those are just the two on the y-axis. But, what is more interesting is when $1 + 2k\sin\theta = 0$, i.e. if $\sin\theta = -\frac{1}{2k}$. Notice this can't happen if $k < \frac{1}{2}$, because $\frac{1}{2k} > 1$. That's our $\frac{1}{2}$ boundary. We know already from above that the point is when it touches the origin, which happens if $k = 1$, and that the loops come when $r < 0$. I think that's the complete analysis. I look forward to seeing what you find.

For +2 on this assignment - anyone in any order not only the first one - may send me their favourite parametric (non-polar) graph and their favourite polar graph. +1 for each. More pretty pictures for all! But, make your own, don't steal someone else's favourite - that's mean. Include both the graph and the function for all. The deadline on this and all solutions extra credit is the *start* of our final exam, on Friday.