222 Assignment 6 Solutions

By the way - the arrows do *not* belong. That is an error in the book, it's a difficult one to fix.

§3.2 25 & 27. I think I'm doing 27 first. $y' = \sin^2 y$. So, what is noteworthy here? y' doesn't depend on x (or t). So, it should be the same across each row that has the same y values. Values are never negative. And values cycle, starting a zero for $y = 0$, the x-axis and increasing to one as we go up (and down) then back to zero. So, that's some of what to look for. Looking at our options, zero on the x-xis is common, but only option (d) is constant across rows. So, apparently all we needed to see was that this didn't depend on t. That's pretty simple. It's also the only one with no negative slopes. Ok, so that was easy. You might say ... some of these look as if the slope is greater than 1, especially on the top and the bottom. Notice carefully that the scale is larger on the x-coordinates than the y coordinates, so this compressed in x which makes it look larger. So, what about 25? $y' = -t\cos y$. Ok, one thing I see here is that for $t = 0$ the slope is zero. So, zero on the y-axis. And *not* always zero on the x-axis, because for $t = 1, y = 0$ we get $y' = -1$. That alone narrows our choices to (e), because all the others are constantly zero on the x-axis. Of note, something interesting is happening around $y = 1.5$, which is reasonable since $\pi/2$ is close to 1.5, where $\cos(\pi/2) = 0$. This looks good. We didn't need to notice much to choose from these options.

§3.2 32. Because we want 5 steps to go from 0 to 1, we will take $h = \Delta x = 0.2$. We start at $(0, 3)$. We are given that $y' = y + t^2$. So $y'(0, 3) = 3$, and $y(0.2) = 3(0.2) + 3 = 3.6$. So, our second point is $(0.2, 3.6)$. We use this point to find our new derivative, $y'(0.2, 3.6) = 3.6 + (0.2)^2 = 3.64$. And we then take another step, $y(0.4) = 3.64(0.2) + 3.6 = 4.328$. Now we switch over to the computer and let it do some work for us. For it we will call our equation $y' = y + x^2$, set the start and Δx. The machine tells us our second points is $(0.4, 4.33)$, so apparently it is rounding, which adds yet another source of error to the approximation. We now see what the next 3 steps are, leading to $y(1)$. The machine says the next points are $(0.6, 5.23)$, $(0.8, 6.34)$ and finally $(1, 7.74)$. From the graph it looks as if our approximation is a bit below the solution. In the problem we are told that the exact solution is $y = 5e^t - 2 - t^2 - 2t$ (I sincerely have no idea how one finds that, but differential equations are the furthest from my speciality - it does look easy to check. I'll give +1 to the first person who emails me a check that this solution works.). This is easy to compute at $t = 1$, we get $5e - 2 - 1 - 2 = 5e - 5$. This is about equal to 8.5914 (a little more). It looks as if Euler's method is having trouble keeping up, which is natural with such a rapidly growing function.

§3.3 8. $\frac{dy}{dt} = y\cos(3t + 2)$. We separate variables, $\frac{1}{y}\frac{dy}{dt} = \cos(3t + 2)$. Now we integrate both sides with respect to t to get $\int \frac{1}{y}\frac{dy}{dt}dt = \int \cos(3t + 2)dt$. The first one collapses by substitution as always and we get $\int \frac{1}{y}dy = \int \cos(3t + 2)dt$. So, now we integrate to get $\ln y = \frac{1}{3}\sin(3t + 2) + C$. Lastly we solve for y by exponentiating both sides. This produces $y = e^{\frac{1}{3}\sin(3t+2)+C} = e^{\frac{1}{3}\sin(3t+2)}e^C = ke^{\frac{1}{3}\sin(3t+2)}$ for a constant k (since C wasn't very important). In hindsight, this solution should look good with the original differential equation.

§3.3 48. Leaves! Last year I worked with two groups and we came up with a fascinating approximation. Independently we both found that there are about two billion (that's 2,000,000,000) leaves in the village of Geneseo every year. Wow. No wonder it takes a lot of work to clean them up each autumn. Anyway, they do decay, but not immediately. I'm going to try using L as a variable for leaf litter, (capital to avoid issues with the lower case as a variable). I'm using units of g/cm^2 for L. $\frac{dL}{dt} = 2 - 0.9L$. Notice that the rate of change is that it grows by 2 g/cm^2 each year, while 90% of what is there (i.e. of L decays). We now can solve this equation; we separate variables, to get $\frac{1}{2-0.9L}\frac{dL}{dt} = 1$. We integrate both sides, collapsing the left to get $\int \frac{1}{2-0.9L}dL = \int dt$ and hence $-\frac{1}{0.9}\ln(2 - 0.9L) = t + C$. Exponentiating and renaming the constant we get $2 - 0.9L = ke^{-0.9t}$. We may then solve for L to get $L = \frac{2-ke^{-0.9t}}{0.9}$. We are asked to assume that at $t = 0$, $L = 0$, substituting both gives: $0 = \frac{2-k}{0.9}$, so $k = 2$, hence $L = \frac{2-2e^{-0.9t}}{0.9}$. Does this approach a steady value? Yes, if we take the limit of this as t goes to infinity we get $\frac{2}{0.9}$. So, after many years there will be around $2\frac{2}{9}$ g/cm^2 before the leaves fall again. (Sure, if someone different from the check above sends me full work checking this differential equation solution, I will again give +1.).

Population project #11. For comparison, I will try my hand at doing this. Answers vary. I'm looking at a scatter plot of the data. I'm going to use 1900, 1970 and 2010 as my data points. From what most of you said, it sounded as if you were taking 1790 as 0 and working in units of years. So, I will be do that. So, I am using $p(110) = 76212168$, $p(180) = 203302031$ and $p(220) = 308745538$. I think the a, b, c form is going to be easier to work with, but first dividing all by e^{at} so that there's only one a, so off I go: $76212168 = \frac{b}{c+e^{-110a}}$, $203302031 = \frac{b}{c+e^{-180a}}$ and $308745538 = \frac{b}{c+e^{-220a}}$. I will divide the first two by each other to and same with the last two to get:

$$2.66758 = \frac{c + e^{-110a}}{c + e^{-180a}} \qquad\qquad 1.51865 = \frac{c + e^{-180a}}{c + e^{-220a}}$$

Notice that we've already eliminated b. Also notice that this is all modeling, so my calculator and I decide to round after 5 decimal places. That's a step. We can multiply out the fractions, and gather the c terms to one side to get:

$$1.66758c = e^{-110a} - 2.66758e^{-180a} \qquad\qquad 0.51865c = e^{-180a} - 1.51865e^{-220a}$$

Next we can divide these two equations by each other, to remove c entirely. We will divide the first by the second, and get an equation only involving a:

$$3.21176 = \frac{e^{-110a} - 2.66758e^{-180a}}{e^{-180a} - 1.51865e^{-220a}}$$

Multiply out the fraction again to get

$$3.21176e^{-180a} - 4.87754e^{-220a} = e^{-110a} - 2.66758e^{-180a}$$

Gathering on one side

$$4.87754e^{-220a} - 5.87934e^{-180a} + e^{-110a} = 0$$

And dividing out e^{-110a}

$$4.87754e^{-110a} - 5.87934e^{-70a} + 1 = 0$$

Either plotting or using a numerical solver gives $a = 0.017825$. Now, going back to $1.66758c = e^{-110(0.017825)} - 2.66758e^{-180(0.017825)}$ gives $c = 0.019753$, and now we can go back to one of the first equations, say $76212168 = \frac{b}{0.019753+e^{-110(0.017825)}}$ to get $b = 12232600$.

Ok, ok, that gives us $p(t) = \frac{12232600}{0.019753+e^{-0.017825t}}$. As $t \to \infty$, we get a limiting capacity of $619,277,000$

This model gives $p(230) = 336,710,000$ for 2020, which compares to the actual value of $329,500,000$ and gives $p(260) = 415,172,000$ for 2050. Taking the second derivative and finding where it is zero gives an inflection at $t = 220$, or 2010, which looks believable from the graph originally.