222 Assignment 5 Solutions

§4.5 23 Our task is to decide if there is conditional, absolute, or no convergence. Off we go. $\sum_{n=1}^{\infty}(-1)^{n+1}(\ln(n+1)-\ln n)$. This is (obviously) an alternating series. The absolute value of the terms is $a_n = \ln(n+1)-\ln n = \ln(\frac{n+1}{n})$. $\lim_{n\to\infty}\frac{n+1}{n} = 1$ so $\lim_{n\to\infty}\ln(\frac{n+1}{n}) = 0$. That's one step toward convergence. $\frac{n+1}{n} = 1+\frac{1}{n}$ which is decreasing as n increases because the denominator is increasing. Same goes for the logarithm of it, so this sequence is decreasing to zero. By the alternating series test, this series converges, conditionally at least. Now we turn to absolute, where the question becomes: does $\sum_{n=1}^{\infty}(\ln(n+1)-\ln n)$ converge? I wonder how to settle this. It's positive and decreasing and continuous, but I'd rather not integrate it. Graphing it looks *very* close to the harmonic series, so I'm going to try limit comparison: $\lim_{n\to\infty}\frac{\ln(1+1/n)}{\frac{1}{n}}$. I'm going to let $x = 1/n$ to switch to a limit to zero: $\lim_{x\to 0}\frac{\ln(1+x)}{x}$. That looks pretty manageable. Bernoulli's rule is happy for this one, and this limit equals $\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}$, and happily this limit goes to 1. So, limit comparison worked fabulously. The absolute value series does the same thing as the harmonic series, which is diverges. *So*, we have that the series converges conditionally - i.e. it needs the negatives to converge.

Oh, there's another way, you may like this better. For the absolute convergence, we're considering $\sum_{n=1}^{\infty}(\ln(n+1)-\ln n)$. This equals $\ln 2-\ln 1+\ln 3-\ln 2+\ln 4-\ln 3+\ln 5-\ln 4+\cdots = \lim_{n\to\infty}\ln(n+1)-\ln 1$. This limit diverges, so the sum diverges.

§4.5 41 Is this statement true or false: If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges conditionally but not absolutely, then b_n does not tend to zero. This is false. If it converges at *all* then the terms *must* go to zero. That is the divergence test. The directions say "If the statement is false, provide an example in which it is false." That is a bit silly since this statement is always false, any series that converges have the terms tend to zero. So, we *only* need a conditionally convergence series (but we do need one). The standard example is the alternating harmonic: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. So, there.

§4.6 28 Ok, $\sum_{n=1}^{\infty}(n^{1/n}-1)^n$ is a pretty clear example of a rare time when we want to use the root test. The ratio test would be a mess. The root test is, however, very simple, we check: $\lim_{n\to\infty}(n^{1/n}-1$ and recall since we've done it enough that the first limit goes to 1, so the whole goes to 0, which is less than one. So, this converges. Done. Simple.

§4.6 30 I *think* this question is a mistake. Here's why: for +1, send me the proof using the divergence test that this diverges, for +2 send me an analysis of the similar problem $\sum_{n=1}^{\infty}\frac{(2^n)^2}{n^n n!}$. I think that problem is probably what they meant, it fits the ratio test better and wouldn't be settled using the divergence test (for +1 show me why the divergence test *doesn't* answer the second question). But, I'm sticking with the question nonetheless. It's still ok practice, if a bit misdirected. Here's the original $\sum_{n=1}^{\infty}\frac{2^{n^2}}{n^n n!}$. We're told to use the ratio test (perhaps foolishly), so we do: $\lim_{n\to\infty}\frac{\frac{2^{(n+1)^2}}{(n+1)^{n+1}(n+1)!}}{\frac{2^{n^2}}{n^n n!}} = \lim_{n\to\infty}\frac{2^{(n+1)^2}n^n n!}{(n+1)^{n+1}(n+1)!2^{n^2}} = \lim_{n\to\infty}\frac{2^{n^2+2n+1}n^n n!}{(n+1)^{n+1}(n+1)!2^{n^2}}$. We cancel 2^{n^2} and the factorials to get $\lim_{n\to\infty}\frac{2^{2n+1}n^n}{(n+1)^{n+1}(n+1)}$. I'm claiming that we've seen some of this before a few times: $\lim_{n\to\infty}(\frac{n}{n+1})^n = \lim_{n\to\infty}(\frac{1}{1+1/n})^n = \frac{1}{e}$. Pulling that out of the original limit, we're left with $\lim_{n\to\infty}\frac{2^{2n+1}}{(n+1)^2}$. This diverges to infinity. If you want you can run Bernoulli's rule twice, the denominator will differentiate to a constant, but the numerator will keep having powers of 2. And infinity times $\frac{1}{e}$ is infinite, which is far from less than 1. So, this series diverges, as we could've known from the beginning. Decent practice, but probably not what was intended, but it leads to some good extra questions. I feel ok about it.

§5.1.18 We want to find where $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$ converges. As always, we start with the ratio test. $\lim_{k\to\infty} \frac{\frac{(k+1)^e x^{k+1}}{e^{k+1}}}{\frac{k^e x^k}{e^k}} = \lim_{k\to\infty} \frac{x}{e}\left(\frac{k+1}{k}\right)^e$. This converges to $\frac{x}{e}$. $\left|\frac{x}{e}\right| < 1$ when $|x| < e$. So the radius of convergence is e. Does it converge at the endpoints? Check $x = e$ and $x = -e$. At $x = e$ we have $\sum_{k=1}^{\infty} \frac{k^e e^k}{e^k}$ which obviously diverges by the divergence test. The other enpoint $x = -e$ is much the same $\sum_{k=1}^{\infty} \frac{k^e(-1)^k e^k}{e^k}$. Therefore this converges on $(-e, e)$. I think this question would have been more interesting with $\sum_{k=1}^{\infty} \frac{x^k}{k^e e^k}$. +1 for anyone who sends me that solution.

§5.1.42 I'm going to focus on the problem as I changed it, until the end, where I will explain why. We're looking for a power series for $f(x) = \frac{(x-2)^2}{5-4x+x^2}$ centred at $a = 2$. I admit, putting the numerator like that gave up the game. I could have written it as $f(x) = \frac{4-4x+x^2}{5-4x+x^2}$. In any case, this is centred at $a = 2$. Because of that we want it all in terms of $(x-2)$. We're pretty close and the distributed form gives even more clues. Looking at what we have in both forms we can push toward $f(x) = \frac{(x-2)^2}{1+(x-2)^2}$ which looks pretty close to $\frac{1}{1-y}$ which is our geometric series. So, if we let $y = -(x-2)^2$ and then multiply by $(x-2)^2$, we get $f(x)$. We can view this as $f(x) = (x-2)^2 \frac{1}{1-(-(x-2)^2)}$. But, $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$, so substituting, $f(x) = (x-2)^2 \sum_{n=0}^{\infty} (-(x-2))^n$. Now manipulating(factoring out the negative, and gathering the $x-2$ factors) we get: $f(x) = \sum_{n=0}^{\infty} (-1)^n (x-2)^{2n+2}$. There's our series. Now, what about convergence? We start with the ratio test, as always: $\left|\lim_{n\to\infty} \frac{(-1)^{n+1}(x-2)^{2n+4}}{(-1)^n(x-2)^{2n+2}}\right| < 1$. This limit simplifies pretty quickly to $\left|(x-2)^2\right| < 1$ Which happens for $1 < x < 3$. What about endpoints? If you put in $x = 3$ we get: $f(3) = \sum_{n=0}^{\infty} (-1)^n (1)^{2n+2}$ which obviously diverges since we keep adding ± 1. We get exactly the same thing for $f(1)$. So, this series does not converge at either endpoint. Therefore, the interval of convergence is $(1, 3)$.

All of this was much easier than it would have been with x^2 on the numerator, because then it would need to be shifted in terms of $(x-2)$. It's a long path. I will give +4 to anyone who presents me that solution.

§5.2 30. Remember $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. We are suggested to compute $f'(\frac{1}{3})$. First $f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$. At $x = \frac{1}{3}$ this equals $f'(\frac{1}{3}) = \sum_{n=1}^{\infty} \frac{n}{3^{n-1}} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4}$. So far so good, and nothing really challenging yet. The question asks to find $\sum_{n=1}^{\infty} \frac{n}{3^n}$. This has one more 3 in the denominator, so this is $\frac{1}{3}$, of the above, i.e. $\frac{3}{4}$. All not bad. By the way, if we started with $f(x) = \sum_{n=0}^{\infty} x^{6n}$, the derivative would be $f'(x) = \sum_{n=0}^{\infty} (6n)x^{6n-1}$, and that's uselessly hopeless. Clearly this was a stray typo. Not even interesting enough to make a new problem out of.

§5.2 46. Let's put our pieces together up to the third degree. This won't match beyond that, because the matching happens in the higher terms. $E(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \ldots$, $E(y) = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 \ldots$, and $E(x+y) = 1 + (x+y) + \frac{1}{2}(x+y)^2 + \frac{1}{6}(x+y)^3 \ldots$ which we will need to distribute to get
$E(x+y) = 1 + (x+y) + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}y^3 \ldots$. Now we compare to multiplying $E(x)E(y) = (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \ldots)(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 \ldots)$
$= 1 + x + y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + xy + \frac{1}{6}x^3 + \frac{1}{6}y^3 + x\frac{1}{2}y^2 + y\frac{1}{2}x^2 + \frac{1}{2}x^2\frac{1}{2}y^2 + x\frac{1}{6}y^3 + y\frac{1}{6}x^3 \ldots$. Notice that term-by-term this is the same as $E(x+y)$ .. up to the third degree. The last four terms that I wrote for $E(x)E(y)$ are of

fourth degree, and they come from the $(x+y)^4$ term. They all do match up ...in the infinite series. And, we see how to multiply infinite series - matching degrees.

§5.4 29. Oh, we didn't use the word "Maclaurin". There's good reason for this. Maclaurin wouldn't really have wanted it. Maclaurin wrote the first calculus book in English, and in it he talked about Taylor series, giving credit to Taylor. Weird people today call Taylor series centred at 0 Maclaurin series. That's silly. It is widespread, but still silly. Anyway, this question should (happily) be short. We just did this above, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So, $e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$. And we can multiply this all by x to get our final answer $xe^{3x} = x \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{n+1}}{n!}$. We could've done this after our first lab. We've come full circle.

§5.4 57-59. What is $\sinh x$? It is hyperbolic sine. It is what would happen if you used hyperbolas instead of circles for defining trigonometry (oh, in case you missed that - we definite trigonometry by the unit circle, not really by triangles, they appear in the unit circle, but the circle matters more). The hyperbolic cosine, $\cosh x$, curve is the equation of a hanging cable. Anyway, that's a bit of context. I'm following the way suggested in the answers for 58, although you could take derivatives, which is pretty nice also. I think this one is nice starting with dots. $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \dots$. And $e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \dots$. If we add these two together, the odd terms cancel, and the even terms double. Then dividing by two we are just left with the even terms, hence $\frac{e^x+e^{-x}}{2} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x$. I guess I did 58 first.

If we subtract the two series then the even terms cancel, and because $1-(-1)=2$ the odd terms double. Dividing by two we are just left with the odd terms, hence $\frac{e^x-e^{-x}}{2} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$

If we take the derivative $\frac{d}{dx}\sinh x = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x$.

These two functions are very nice together for the following reasons: $\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x+e^{-x}}{2} = \frac{e^x-e^{-x}}{2} = \sinh x$ and $\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x-e^{-x}}{2} = \frac{e^x+e^{-x}}{2} = \cosh x$. This is another way to explain everything here.