222 Assignment 4 Solutions

§5.3 15 Start with $\sin t \simeq t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$. Substitute πt in for t to get $\sin \pi t \simeq \pi t - \frac{(\pi t)^3}{6} + \frac{(\pi t)^5}{120} - \frac{(\pi t)^7}{5040}$. Divide all by πt to get $\frac{\sin \pi t}{\pi t} \simeq 1 - \frac{(\pi t)^2}{6} + \frac{(\pi t)^4}{120} - \frac{(\pi t)^6}{5040}$. Now integrate $\int \frac{\sin \pi t}{\pi t} dt = \simeq t - \frac{\pi^2 t^3}{18} + \frac{\pi^4 t^5}{600} - \frac{\pi^6 t^7}{35280}$. Finally evaluate from 0 to 1. This is simple, zero makes all zero and one changes all ts to 1. So we get a final approximation of $1 - \frac{\pi^2}{18} + \frac{\pi^4}{600} - \frac{\pi^6}{35280}$. The question does not say to estimate this approximation.

§5.3 25 We're recentring x^4 at $a = -1$. To do so, we need some derivatives: $f'(x) = 4x^3, f'''(x) = 12x^2, f'''(x) = 24x, f''''(x) = 24$ (all higher derivatives are zero). The values at -1 are, respectively, starting with the function,, $1, -4, 12, -24, 24, 0, \ldots$. We then assemble this into the polynomial to get $x^4 = 1 - 4(x+1) + 6(x+1)^2 - 4(x+1) + 1$. Of note, we can get this by computing $[(x+1)-1]^4$ and protecting the $(x+1)$ part.

§4.1 36 We're considering $n^{-1/n}$. The question asks about bounded, about monotone, increasing, and decreasing. Before I get into it, I want to rewrite as $\frac{1}{\sqrt[n]{n}}$. Roots of positive natural numbers will stay positive, so this is bounded below by 0. Because $1^n = 1$, $\sqrt[n]{n} > 1$ for all $n \geq 1$, because of this, $\frac{1}{\sqrt[n]{n}} < 1$ for all $n \geq 1$. Hence this sequence is bounded. To assess increasing or decreasing we will try to take a derivative. That's not as easy as it seems, since this requires logarithmic differentiation, but it's a good chance to remind ourselves. For derivatives we'll switch to $y = x^{-1/x}$. Then take logs of both sides to get the exponent down $\ln y = -\frac{\ln x}{x}$. Differentiate both sides, giving $\frac{y'}{y} = \frac{\ln x - 1}{x^2}$, and hence $y' = x^{-1/x}\frac{\ln x - 1}{x^2}$. This is positive as long as $\ln x > 1$, which happens if $x > e$, or surely if $n > 3$, as was given (for that reason). So, this is increasing. The question doesn't ask - it therefore must converge. It also doesn't ask what to. I really do want to give someone points who reads solutions, and I've given *none* so far, so I try again. +1 for the first person to present the work needed to show what it converges to.

§4.1 42 $\frac{n!}{n^n} \geq 0$, so that's the easy part. Now we need something that it is smaller than, ideally that also goes to zero. Notice: $\frac{n!}{n^n} = \frac{123\cdots(n-1)(n)}{nnn\cdots nn} = \frac{1}{n}\frac{23\cdots(n-1)(n)}{nn\cdots nn} < \frac{1}{n}$ because the second fraction is less than one (at least as n gets large). But, $\lim_{n\to\infty} \frac{1}{n} = 0$. So $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ because it is trapped between 0 and 0.

§4.2 16 The harmonic series is small, but not small enough. This is smaller. We're looking at $\sum_{n=1}^{\infty} \frac{1}{2n+1} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \frac{1}{27} + \cdots$. We are told to use partial sums, so despite really wanting to use the integral test (for which this is easy ... hey, +1 for the first person to email me how to use the integral test here. If no one does this, then no one is reading solutions at all). We're following the ideas from the lab proof. Look at that one again. This time we will group in powers of three (because it's smaller): $(\frac{1}{3}) + (\frac{1}{5} + \frac{1}{7} + \frac{1}{9}) + (\frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \frac{1}{27}) + \cdots$ Each of the fractions in a group is larger than the *last* fraction in that group, and so we see this is $> (\frac{1}{3}) + (\frac{1}{9} + \frac{1}{9} + \frac{1}{9}) + (\frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}) + \cdots$. Now we can groups the group together to get $\frac{1}{3} + \frac{3}{9} + \frac{9}{27} + \cdots$. This continues for each group of a power of 3, so there are next $\frac{27}{81}$. Each of them is equal to $\frac{1}{3}$. And there are therefore an endless number of $\frac{1}{3}$s added together, which then diverges.

Here's a version that is more like the first proof we saw in the worksheet (I learned this version from students in our class). We're looking at $\sum_{n=1}^{\infty} \frac{1}{2n+1} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \cdots$. Suppose this converges to a sum, S. Make the a smaller series by pushing the first term down to the second, and continue that pattern for the alternate terms, so $S > \frac{1}{5} + \frac{1}{5} + \frac{1}{9} + \frac{1}{9} + \frac{1}{13} + \frac{1}{13} + \frac{1}{17} + \frac{1}{17} + \frac{1}{21} + \frac{1}{21} + \frac{1}{25} + \frac{1}{25} + \cdots$. It's unconventional, but true that $\frac{1}{5} + \frac{1}{5} = \frac{2}{5} = \frac{1}{2.5}$. Using similar reasoning we find $S > \frac{1}{2.5} + \frac{1}{4.5} + \frac{1}{6.5} + \frac{1}{8.5} + \frac{1}{10.5} + \frac{1}{12.5} + \cdots$. Remembering what S is supposedly equal to, we have $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \cdots > \frac{1}{2.5} + \frac{1}{4.5} + \frac{1}{6.5} + \frac{1}{8.5} + \frac{1}{10.5} + \frac{1}{12.5} + \cdots$, but the terms in the first sum are one by one *smaller* not larger, because the denominators are larger. This makes no sense, so apparently the series cannot converge.

§4.2 48 Clearly this should remind you of $\sum \frac{1}{n(n+1)}$. So we will try doing the same things with $\sum \frac{1}{n(n+N)}$. We start out trying to separate with partial fractions $\frac{1}{n(n+N)} = \frac{A}{n} + \frac{B}{n+N}$. Clear fractions to get $1 = A(n+N) + Bn$. If $n = 0$, we get $1 = AN$, so $A = \frac{1}{N}$. If $n = -N$ we get $1 = -BN$, so $B = -\frac{1}{N}$.

So, $\frac{1}{n(n+N)} = \frac{1}{N}(\frac{1}{n} - \frac{1}{n+N})$. So, $\sum_{n=1}^{\infty} \frac{1}{n(n+N)} = \frac{1}{N}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+N}$. Let's write out some terms of this ... $\frac{1}{N}(\frac{1}{1} - \frac{1}{N+1} + \frac{1}{2} - \frac{1}{N+2} + \frac{1}{3} - \frac{1}{N+3} + \ldots \frac{1}{N} - \frac{1}{2N} + \frac{1}{N+1} - \frac{1}{2N+1} + \ldots)$. Notice at this point that some fractions start canceling. The "rest" will all cancel, and we will be left with $\frac{1}{N}(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{N})$ (note that this *ends*). We can write this in summation form, but I don't know if that's much better. Here it is $\frac{1}{N}\sum_{k=1}^{N} \frac{1}{k}$. I'm happy for either answer. I don't believe this can be better. I will give +5 or more for anyone who shows me otherwise. I do believe that taking this further may be a research project, but I think it is unknown in any useful way.

§4.3 7 The divergence test does not apply because $\lim_{n\to\infty} a_n = 0$. Here's the work: $\frac{2^n+3^n}{10^{n/2}} = \frac{2^n+3^n}{\sqrt{10}^n} = \frac{2^n/\sqrt{10}^n+3^n/\sqrt{10}^n}{\sqrt{10}^n/\sqrt{10}^n}$. Notice that $\sqrt{10} > 3$, so when we take limits the top sequences go to zero, while the denominator goes to one. Hence: $\lim_{n\to\infty} \frac{2^n+3^n}{10^{n/2}} = \lim_{n\to\infty} \frac{2^n/\sqrt{10}^n+3^n/\sqrt{10}^n}{1} = \frac{0+0}{1} = 0$, as claimed. This tells us nothing. The case can be settled by deft use of basic comparison and the ratio test. I will give +2 to the first person who sends me that.

§4.3 45 $R_N < \int_N^{\infty} \frac{1}{n^6} dn = \frac{1}{5N^5}$. We want error $< 10^{-6}$, so $\frac{1}{5N^5} < 10^{-6}, 10^6 < 5N^5$, or $N > 10\sqrt[5]{2} > 11.48$, so $N = 12$ will work. Let's go back to MathisFun ... it says $\sum_{n=1}^{12} \frac{1}{n^6} = 1.017342411813338$. The error from the approximation in the book is $6.48187 \times 10^{-7} < 10^{-6}$, as desired. As a fascinating aside, notice again, like $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the actual sum is a again a curious faction with a power of π. These p-series have surprising sums. Euler derived many of these, up to $\sum_{n=1}^{\infty} \frac{1}{n^{26}}$. The first person who emails me the exact value of that sum, I will give +1. You could just look that up!! Perhaps even more surprising *no one knows* the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^3}$. The even ones are all π to a power divided by a natural number, but the harmonic diverges, and we don't know what the others are. In 1978 it was proven that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational (this is in my lifetime, and compared to Euler's work from 1734 this is recent history). We don't even know that much for the other odd powers.

§4.4 24. There is accidentally a hint here. Fine. It's hiding on the next page. The limit comparison part is not surprising. The $\frac{1}{n^3}$ part may be. Before we do what we're told, let's see why the obvious thing *doesn't* work. I would've started with $\lim_{n\to\infty} \frac{\frac{1}{n}-\sin(\frac{1}{n})}{\frac{1}{n}}$. We'll use a tactic that we will use in our real work ... as n goes to infinity, $1/n$ goes to zero. So, we may use $x = 1/n$ and all our limits to zero instead. That makes it nicer, for both versions. So, $\lim_{n\to\infty} \frac{\frac{1}{n}-\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{x\to 0} \frac{x-\sin x}{x} = \lim_{x\to 0} 1 - \frac{\sin x}{x} = 1 - 1 = 0$. Zero is not a great result for limit comparison. It tells us that eventually the numerator is smaller. Unfortunately, the denominator is the harmonic series which (of course) diverges.

So, now to try what we were told to do: $\lim_{n\to\infty} \frac{\frac{1}{n}-\sin(\frac{1}{n})}{\frac{1}{n^3}}$. We may use $x = 1/n$ and our limits to zero instead. So, $\lim_{n\to\infty} \frac{\frac{1}{n}-\sin(\frac{1}{n})}{\frac{1}{n^3}} = \lim_{x\to 0} \frac{x-\sin x}{x^3}$. Unlike the first attempt, we cannot split this into known pieces, but it is a Bernoulli's rule form, so we try attacking it with derivatives. It will give in, it just takes some time. We know if we differentiate the denominator enough we will win. Here we go: $= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$, which is nonzero and finite, therefore the two are comparable. Because $\sum \frac{1}{n^3}$ converges, so does our series.

§4.4 29. Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? For which p? Ok, to start, if $p \leq 0$, it diverges by the divergence test, as the terms do not go to zero. That is probably something you ignored, and that's plenty ok. It's not the point. I am looking forward to seeing how people work on this problem. Can we do it by basic comparison? Can we show that for all p, $(\ln n)^p < n$, at least "eventually", i.e. for n large enough? I don't see a nice way to do that. What I do see is using the limit comparison. If no one hands it in with a viable solution without using limits, I will give +4 for the first who writes up a correct solution without using limits. Here's my point of view ... if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then b_n is much faster than a_n, and so by limit comparison, if a_n diverges, then b_n is definitely going to diverge. For this approach, I will be comparing to $\frac{1}{n}$. So, I am considering the limit $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(\ln n)^p}}$. The top is the terms of the harmonic series which diverges. If I show this is zero, then I will know that our series diverges for all p. As a first step, we can invert both fractions to get ... $\lim_{n\to\infty} \frac{(\ln n)^p}{n}$ (at least it's easier to read). Notice that this is now an $\frac{\infty}{\infty}$ form so we may use Bernoulli's rule. The derivative of the denominator is just 1, so that's nice. The derivative of the numerator is itself a fraction, it is $\frac{p(\ln n)^{p-1}}{n}$. The similarity is interesting. If we apply Bernoulli's rule again, taking derivatives of numerator and denominator we get $\frac{p(p-1)(\ln n)^{p-2}}{n}$. So, um, how does it end? We keep taking derivatives, and this continues. *If* p was a natural number, we will eventually reach 0, at that point we will have $\frac{p!}{n}$. But wait, we lost the limit, this is actually $\lim_{n\to\infty} \frac{p!}{n} = 0$ (because the numerator is a number and the denominator is going to infinity). That's great, and mostly settles the story (because our terms are eventually much larger than the harmonic series which diverges), but what if p is not an integer? Probably this is not the intent of the question, but it's actually not bad given what we've done. Notice that if $p < q$, then $\frac{1}{(\ln n)^p} > \frac{1}{(\ln n)^q}$. We know our series diverges for all integer values of p. If p is not an integer, there is an integer q that is bigger than it (the next one would be the natural choice, but we don't need to be so careful), then $\frac{1}{(\ln n)^p} > \frac{1}{(\ln n)^q}$, and since we know that the second series diverges for all integers, the series with p must also diverge, no matter what p is.