222 Assignment 2 Solutions

§1.3 43

There's an interesting issue here ... these curves intersect three times. Because of that we could mean the two regions. I'm not sure if that is what is meant. It wouldn't be much more difficult if it were. I will presume they mean just the first intersection. But, I will set up the second one at the very end. Since I don't want to solve $x = \sin(\pi y^2)$ for y, I will work in terms of y. The intersections are at 0 (I hope that's obvious), for one. The second one is tricky. We want $\sin(\pi y^2) = \sqrt{2}y$. The clue is the $\sqrt{2}$. We like to hope that you recall that $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. Notice that if $y = \frac{1}{2}$, this makes both sides work out. That is our second intersection. Beyond that remember $\sin(\frac{\pi}{2}) = 1 = \frac{\sqrt{2}}{\sqrt{2}}$. See that this is satsfied if $y = \frac{1}{\sqrt{2}}$. Notice neither of these can really be "solved", but instead just recognised. So ... now to set up our integrals. Again because I don't want to invert the sine equation, I am working in terms of y which gives us horizontal segments. Horizontal segments rotated around the x-axis produces cylinders. So we use $2\pi rh$. The radius is just y since it is an axis. The height is the difference of the x-coordinates. The bigger one is $\sqrt{2}y$ and the smaller one is $\sin(\pi y^2)$. So the area of the cylinder is $2\pi y(\sqrt{2}y - \sin(\pi y^2)$. The range of y values for the first region is from 0 to $\frac{1}{2}$. So, our integral for volume is $\int_0^{\frac{1}{2}} 2\pi y(\sqrt{2}y - \sin(\pi y^2))dy$. Wow, that's a lot of work for setup. Now to integrate. Let's distribute, break into two integrals, and pull out constants to get: $2\sqrt{2}\pi \int_0^{\frac{1}{2}} y^2 dy - \int_0^{\frac{1}{2}} 2\pi y \sin(\pi y^2)dy$. I've tried to leave constants where they will be useful. This integrates to $[2\sqrt{2}\pi\frac{1}{3}y^3 + \cos(\pi y^2)|_0^{\frac{1}{2}}$ (notice the substitution in the second integral). After evaluating we get $\pi\frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{2} - 1$. Remember at the beginning there were two regions? If you did both of them on the problem set I will record 5/4 again (I like doing that). And, if you're just reading this now and you're the first person to email me the correct set up and work for the second integral, I will record +1 for you. I like people reading solutions. This extra is not that difficult, now that all the above work is there. Even if you didn't do the above work.

§1.3 49 We're rotating around $x = 4$, which is a vertical line. We are told to use shells, i.e. cylinders, so we must use a vertical segments, which means we're working in terms of x. We're using the circle $x^2 + y^2 = 4$ which is centred at the origin and has radius 2. Therefore the range of x values is $[-2, 2]$. The distance from $x = 4$ is $4 - x$ (try with $x = -1$ or $x = 3$ if you want to justify this). The height is from the bottom to the top of the circle, i.e. $\sqrt{4 - x^2} - -\sqrt{4 - x^2} = 2\sqrt{4 - x^2}$. So the area of each cylinder is $2\pi(4 - x)2\sqrt{4 - x^2}$. Hence the integral for the volume is $\int_{-2}^{2} 2\pi(4 - x)2\sqrt{4 - x^2}dx$. We can distribute this to two integrals: $16\pi \int_{-2}^{2} \sqrt{4 - x^2}dx - 8\pi \int_{-2}^{2} 2x\sqrt{4 - x^2}dx$. The first would be difficult, but we have some inside information, it tells us the area of the top half off a circle of radius 2. We know that to be $\frac{1}{2}\pi 2^2 = 2\pi$. The second integral is a bit surprising. If you think of the graph of the function, notice that it is above the x-axis for $[0, 2]$ and below for $[-2, 0]$. Those areas cancel out. I will work through the details to be clear, but we now see the answer is $16\pi(2\pi) - 8\pi(0) = 32\pi^2$. For the second integral, $\int_{-2}^{2} 2x\sqrt{4 - x^2}dx$, let $u = 4 - x^2$, $du = -2x$, so we have $\int 2x\sqrt{4 - x^2}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(4 - x^2)^{3/2} + C$, so $\int_{-2}^{2} 2x\sqrt{4 - x^2}dx = [\frac{2}{3}(4 - x^2)^{3/2}|_{-2}^{2} = \frac{2}{3}(0 - 0) = 0$, as claimed before. It's not only 0, it's $0 - 0$.

§1.4 44 The cylinder is a cylinder. It has area $2\pi rh = 2\pi(1/4)(1/3)$ in^2 $= \frac{\pi}{6}$ in^2. The sphere work seems very familiar from class. There are annoying fractions, which makes writing it difficult, but not conceptually difficult. Remember from our work in class - this works out nicely in the end. Don't give up. Here we go. The circle is $x^2 + y^2 = \frac{1}{4}$. So, the function is $\sqrt{\frac{1}{4} - x^2}$. The limits are $-\sqrt{\frac{3}{16}} = -\frac{\sqrt{3}}{4}$ and $\frac{1}{2}$. Here's a nice idea, and something to remember - numbers are annoying and variables are nice. So, I'm going to do this with variables and not numbers, and then use numbers as the end. So, if instead we use $\sqrt{r^2 - x^2}$ from $x = a$ to $x = b$ we get $\int_a^b 2\pi\sqrt{r^2 - x^2}\sqrt{1 + [\frac{-2x}{2\sqrt{r^2 - x^2}}]^2}dx = \int_a^b 2\pi\sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}dx = \int_a^b 2\pi\sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}dx = \int_a^b 2\pi r dx = 2\pi r(b - a)$. This is basically the same as what we did in class, and is a really fascinating fact - that the area of a section of a sphere is just proportional to its height. In any case, from the beginning in our case we just grab the values $2\pi\frac{1}{2}(\frac{1}{2} - -\frac{\sqrt{3}}{4}) = \pi(\frac{2+\sqrt{3}}{4})$. To get the final answer we add the part from the cylinder before, $\frac{\pi}{6}$, to get $\pi(\frac{2}{3} + \frac{\sqrt{3}}{4})$ in^2.

§1.4 50 Ok, we need length of a parabola and length of a line segment from $(0,0)$ to (b^2, b). The second is merely the distance formula, and even simpler because one of the points is the origin, so it is $\sqrt{b^2+b^4} = b\sqrt{b^2+1}$. That's easy work first. So, now for arc-length of parabola, we apply the arc-length formula to $x = y^2$, the good news here is that we can take a square root from 0 to b, i.e. $\int_0^b \sqrt{1+1/(4x)}$. and .. so ... yep, these are problems, as we talked about in class. So, I ditched this problem and replaced it with ...

§1.4 48. We want the arclength of $f(x) = \ln(\sin x)$ from $x = \pi/4$ to $3\pi4$. For the formula we will need $f'(x) = \frac{\cos x}{\sin x}$. And away we substitute: $\int_{\pi/4}^{3\pi/4} \sqrt{1+\frac{\cos^2 x}{\sin^2 x}}dx$. We get a common denominator (because I'm guessing you don't recognise the trig. identity $1+\cot^2 x$), $= \int_{\pi/4}^{3\pi/4} \sqrt{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}dx$ and I hope you recognise that trig. identity $= \int_{\pi/4}^{3\pi/4} \sqrt{\frac{1}{\sin^2 x}}dx = \int_{\pi/4}^{3\pi/4} \frac{1}{\sin x}dx = \int_{\pi/4}^{3\pi/4} \csc x dx$. And at this point either you remember the most obscure trig. integral or you don't. Here is where it comes from: $\int_{\pi/4}^{3\pi/4} \csc x dx = \int_{\pi/4}^{3\pi/4} \csc x (\frac{\csc x + \cot x}{\csc x + \cot x})dx = \int_{\pi/4}^{3\pi/4} \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}dx$. Now, either you do or don't remember the derivatives, they come from the quotient rule: $\frac{d \csc x}{dx} = -\csc x \cot x$ and $\frac{d \cot x}{dx} = -\csc^2 x$. So we are well set up for substitution $u = \csc x + \cot x, du = -\csc^2 x - \csc x \cot x dx$ so we have $-\int \frac{du}{u} = -\ln(u) + C$ and our integral equals $[-\ln(\csc x + \cot x)|_{\pi/4}^{3\pi/4} = \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) = \ln(\frac{\sqrt{2}+1}{\sqrt{2}-1}) = \ln(3+2\sqrt{2})$.

§2.1 45 We have another question with an answer, ok. Let's see if we can get this one to agree. $\int x^2 \sin x dx$. This should be pretty straightforward, parts twice. Let $f = x^2$ and $g' = \sin x dx$ since differentiating f helps. $f' = 2x dx$, $g = -\cos x$. So, $\int x^2 \sin x dx == -x^2 \cos x + \int 2x \cos x dx$. We do it again, $f = 2x$, and $g' = \cos x dx$, so $f' = 2dx$ and $g = \sin x$. So, $\int x^2 \sin x dx = -x^2 \cos x + \int 2x \cos x dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$. And $\int_0^{\pi/2} x^2 \sin x dx = [-x^2 \cos x + 2x \sin x + 2\cos x| \int_0^{\pi/2} = 0 + \pi + 0 - (-0 + 0 + 2) = \pi - 2$. And we even agree.

§2.1 49 We want to find $\int x^n \cos x dx$. We let $f = x^n$, $g' = \cos x dx$. Hence $f' = nx^{n-1}dx$ and $g = \sin x$. So, $\int x^n \cos x dx = x^n \sin x - \int nx^{n-1} \sin x dx$. Wait, that's all? Well, maybe pull the constant out of the integral, to get: So, $\int x^n \cos x dx = x^n \sin x - \int nx^{n-1} \sin x dx = x^n \sin x - n \int x^{n-1} \sin x dx$ (it's ok if you didn't). Wow, that is likely the easiest question all semester.

§2.2 28 $\int \sin ax \cos ax dx$. Wow, I think I said the last one was the easiest. I feel as if this is a competition. We'll use substitution. Let $u = \sin ax$, $du = a \cos ax dx$ so, $\frac{du}{a} = \cos ax dx$. Hence, $\int \sin ax \cos ax dx = \int \frac{1}{a} u du = \frac{u^2}{2a} + C = \frac{sin^2 ax}{2a}$. That's even less impressive than the last one because we could've done that in Calc 1. Hm, is there another way? Yes, there's lots of 'em. Here's a different way: $\sin 2ax = 2 \sin ax \cos ax$, so $\int \sin ax \cos ax dx = \int \frac{\sin 2ax}{2} dx = -\frac{1}{4a} \cos 2ax + C$. They look different. I'll give +2 for the first person to correctly email me why they are the same.

§2.2 32. $\int \sin^2 x \cos^2 x dx$. We'll use $\cos^2 x = \frac{1+cos2x}{2}$ and $\sin^2 x = \frac{1-cos2x}{2}$. So, $\int \sin^2 x \cos^2 x dx = \int (\frac{1-cos2x}{2})(\frac{1+cos2x}{2})dx = \int \frac{1-cos^2(2x)}{4}dx = \int \frac{sin^2(2x)}{4}dx = \int \frac{1-\cos(4x)}{8}dx$. Finally after all that work we can integrate, $= \frac{1}{8}x - \frac{\sin 4x}{16} + C$. That was ok, this feels ... like we haven't done as much real trig integral practice. Well, I hope it comes in §2.3 or in the problems you pick.

§2.3 41. Find the area inside $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Solving for $y = \pm 3\sqrt{1 - \frac{x^2}{4}}$. The range of values are between $x = \pm 2$. Therefore the desired integral is (top - bottom) $= \int_{-2}^{2} 6\sqrt{1 - \left(\frac{x}{2}\right)^2}dx$. So that we can take a square root, we let $\frac{x}{2} = \sin\theta$. From this, $dx = 2\cos\theta d\theta$. If $x = 2, 1 = \sin\theta$, so $\theta = \frac{\pi}{2}$. And if $x = -2, -1 = \sin\theta$, and $\theta = -\frac{\pi}{2}$ We make our substitutions: $= \int_{-\pi/2}^{\pi/2} 6\sqrt{1-\sin^2\theta}(2\cos\theta)d\theta = \int_{-\pi/2}^{\pi/2} 12\cos^2\theta d\theta$. Then we use our identity for $\cos^2\theta$: $= \int_{-\pi/2}^{\pi/2} 6 + 6\cos 2\theta d\theta = [6\theta + \sin(2\theta)|_{-\pi/2}^{\pi/2} = 6\pi$. It isn't much more work to get a general formula for an ellipse with axes a and b. Sure, I'll give +2 for the first person to email that to me.

§2.3 43. Ok, we're computing $\int \frac{dx}{x\sqrt{x^2-1}}$ and we're told to let $x = \sec\theta$, so we do. $dx = \sec\theta\tan\theta d\theta$. Making substitutions gives $\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}}$. Using $\sec^2\theta - 1 = \tan^2\theta$ produces $= \int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\tan^2}} = \int d\theta = \theta + C = \sec^{-1}x + C$. That wasn't too bad, and the second one won't be much worse.

Next we're told to let $x = \csc\theta$, so we do. $dx = -\csc\theta\cot\theta d\theta$. Making substitutions gives $\int -\frac{\csc\theta\cot\theta d\theta}{\csc\theta\sqrt{\csc^2\theta-1}}$. Using $\csc^2\theta - 1 = \cot^2\theta$ produces $= \int -\frac{\csc\theta\cot\theta d\theta}{\csc\theta\sqrt{\cot^2}} = \int -d\theta = -\theta + C = -\csc^{-1}x + C$. So, how do we feel about the two answers. They look different. Here's the important point ... what does "co" mean? It means "of the complement". So, two different co-inverse functions give complementary angles. I am hopeful that you remember that two complementary angles add to $\frac{\pi}{2}$. So, $\sec^1 x + \csc^{-1}x = \frac{\pi}{2}$. As a consequence, $\sec^1 x == \frac{\pi}{2} - \csc^{-1}x$. And that almost explains it, but ... what about the $\frac{\pi}{2}$? It is part of the C. The answers work for any constant, and $\frac{\pi}{2}$ is just another constant. There you have it.

I'm glad that you picked your own problems in Chapter 2 also. I didn't pick very challenging ones, apparently. I hope you can do better than me. Do make sure that you find some to challenge you.