222 Assignment 1 Solutions

§0.6 38. $\int_0^{\pi/3} \frac{\sin x - cos x}{\sin x + \cos x} dx$. This is a pretty routine substitution problem. The derivative of the denominator is basically the numerator. To be more careful, let $u = \sin x + \cos x$, then $du = (\cos x - \sin x)dx$. That is the negative of our numerator so we introduce an extra negative to compensate. $\int \frac{\sin x - cos x}{\sin x + \cos x} dx = -\int \frac{du}{u} = -\ln u + C = -\ln(\sin x + \cos x) + C$. So, $\int_0^{\pi/3} \frac{\sin x - cos x}{\sin x + \cos x} dx = -\ln(\sin x + \cos x)|_0^{\pi/3} = \ln 1 - \ln(\frac{\sqrt{3}+1}{2}) = -\ln(\frac{\sqrt{3}+1}{2})$. I didn't intend to have the first question have an answer to it, but since the answer is somewhat surprising, I'm glad that I did. I'm feeling generous so, I will give 5/4 on this question for anyone who explains the answer in the text. Here is a justification ... $-\ln(\frac{\sqrt{3}+1}{2}) = -\ln(\frac{\sqrt{3}+1}{2}\frac{\sqrt{3}-1}{\sqrt{3}-1}) = -\ln(\frac{2}{2(\sqrt{3}-1)}) = -\ln(\frac{1}{\sqrt{3}-1}) = \ln(\sqrt{3}-1)$, as given in the book.

I also really want to find a way to give credit to people who read solutions. I'm on the lookout for that. This isn't the right place, because here I want to give credit to those who did something extra in what they handed me.

§0.6 65. This one is quite short. The simpler integral is $\int_a^b \frac{1}{x} dx = \ln b - \ln a = \ln\left(\frac{b}{a}\right)$. The other one is about the same, $\int_{ca}^{cb} \frac{1}{x} dx = \ln cb - \ln ca = \ln c + \ln b - \ln c - \ln a = \ln b - \ln a = \ln\left(\frac{b}{a}\right)$. We showed they are the same by computing both and getting the same result.

1. We'll use a common denominator to rewrite it as $\lim_{x\to 0} \frac{\tan 2x + ax + x^2 \sin bx}{x^3}$. Clearly in a $\frac{0}{0}$ form. So we apply l'Hospital, of course, to get $\lim_{x\to 0} \frac{2\sec^2 2x + a + bx^2 \cos bx + 2x \sin bx}{3x^2}$. The denominator is clearly going to zero. To stop the whole thing from being undefined we're going to need the numerator to be zero as well. To accomplish this we'll need $a = -2$. Once we've done that, then we have $\frac{0}{0}$ again. So we apply l'Hospital again to get

$$\lim_{x\to 0} \frac{8\sec^2 2x \tan 2x - x^2 b^2 \sin bx + 4xb \cos bx + 2\sin bx}{6x}.$$

Again the denominator is going to zero. Again we want to make the numerator zero. Carefully examining the numerator, we see it is already. So, we're again in a $\frac{0}{0}$ case. Now, we could take derivatives once more, but let's slow down and think carefully. We're almost done - the denominator is going to be $6 \neq 0$. One last time we want the numerator to be zero and we'll be done. Carefully thinking about what will produce non-zero terms in our derivative we have something that looks like this, *but* ignoring all the terms that will be zero:

$$\lim_{x\to 0} \frac{16\sec^4 2x + \cdots + 6b\cos bx + \cdots}{6}$$

As x goes to zero the numerator goes to $16 + 6b$. To make this, our final step, zero, we need $b = -\frac{8}{3}$. So, looking back, we want $a = -2$ and $b = -\frac{8}{3}$ so that our limit is zero. Well, then. Of note, if we wanted our limit to go to something other than zero, say v, we'd only need to solve $\frac{16+6b}{6} = v$ for b.

2. $\lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right)^x$ is a 1^∞ form. Of course we set it equal to y and take logarithms. That produces $\ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{1}{x^2}\right)$ which is a $\infty \cdot 0$ form. Let $\frac{1}{x} = t$. That changes the problem to $\lim_{t\to 0^+} \frac{\ln(1+t^2)}{t}$. This is a $\frac{0}{0}$ form. Applying Bernoulli produces $\lim t \to 0^+ \frac{2t}{1+t^2} = 0$, so $y = 1$. The obvious graph I can imagine to support this is the original graph which is pretty much necessary, showing the function is heading to 1 on the right of the screen for large values of x (not like 10, but more like a million). The other graph I would think you would like would be the graph after taking logarithm which should approach 0. I will be interested if you collectively have other ideas.

§1.1 38. Find the area bounded by these two equations $x = e^y$ and $y = x - 2$. We'll want them the same, so either $y = \ln x$ and $y = x - 2$ or $x = e^y$ and $x = y + 2$. I'll try the first way first. I don't like solving $\ln x = x - 2$, but I also don't like solving $e^y = y + 2$. On other hand, I prefer integrating e^y over integrating $\ln x$, so I will work in terms of y. The instructions say to use technology to find the intersections to 3 decimal points. I did that and I get $y = -1.841$ and $y = 1.146$. On this interval, $x = y + 2$ is bigger. So our integral is $\int_{-1.841}^{1.146} y+2-e^y dy = \frac{1}{2}y^2+2y-e^y|_{-1.841}^{1.146} = 1.949$. (There's not much to write for putting the numbers in for the limits.

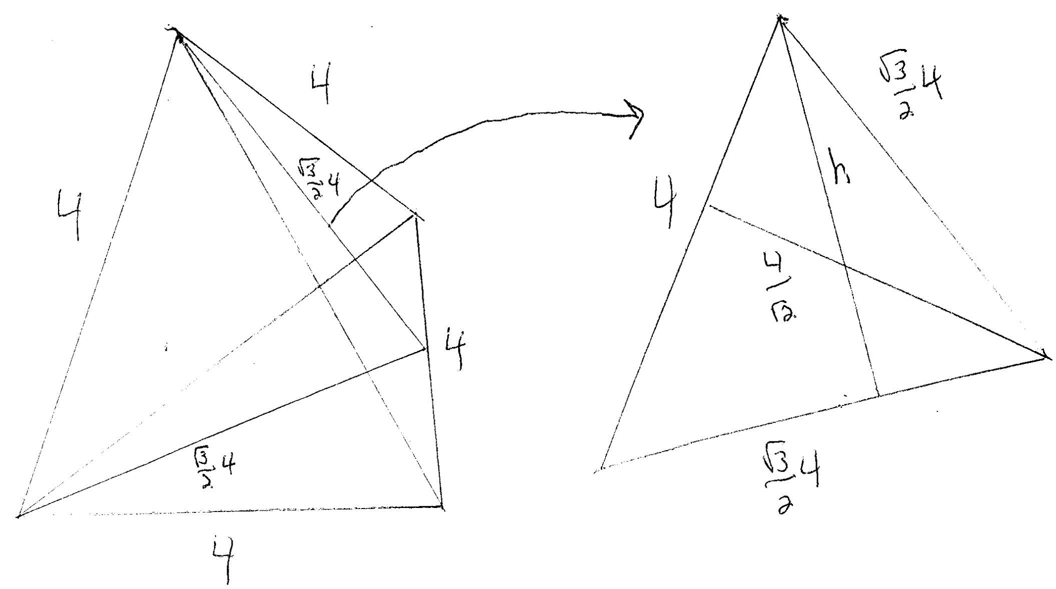
§1.1 56. There are four sectors in this picture. They are each congruent due to quarter turn symmetry in both the square and the circle. Probably you will want to do this as half and multiply by 2 or quarter an multiply by 4. I will show both ways. The top of the square is $y = 1$. That's pretty easy. Because the equation of the unit circle is $x^2 + y^2 = 1^2$, we may solve this for y to get something to use for integration, $y = \pm\sqrt{1-x^2}$. The two parts are the top and the bottom of the circle, each is its own function. We'll just use the top and then multiply. If we use the top half, the integral is $2\int_{-1}^{1} 1-\sqrt{1-x^2}dx$ if we use the first quadrant, the integral is $4\int_0^1 1-\sqrt{1-x^2}dx$. As I mentioned this integral requires Chapter 2 material (and I am confused why it's in §1.1). A machine will tell you that it is approximately 0.858407. Geometrically we can use pre-algebra. The square has area $2^2 = 4$, and the unit circle has area $\pi(1)^2 = \pi$. Therefore the complimentary area is $4-\pi \simeq 0.858407$.

§1.2 15. Ok, in some ways this book is living up to the price, but I still think it's worth it. Remember that each time we have a challenge. There are two ways to interpret this question. The easier one is if the cross sections have a leg of the right isosceles triangle on the base. The area of a right isosceles triangle is $\frac{1}{2}g^2$ where g is the leg of the triangle. So, in this interpretation the cross section areas are $\frac{1}{2}(9-x^2)^2$. The intersections here are $x = \pm 3$, so the integral is $\int_{-3}^{3} \frac{1}{2}(9-x^2)^2 dx$. The left and the right are congruent, so we could use $\int_0^3 (9-x^2)^2 dx$, I like this better both because I can use a 0 limit, and because I can lose the fraction. The next step is to distribute the power $\int_0^3 81-18x^2+x^4 dx = 81x-6x^3+\frac{1}{5}x^5|_0^3 = \frac{648}{5} = 129.6$, exactly. I'm not much a fan of decimals, because they are usually approximations, i.e. wrong, but this one is exact.

The second interpretation is with the hypotenuse of the right isosceles triangle on the base. This triangle has half the area of the former triangle (there are two of this triangle in the above). The area is then $\frac{1}{4}h^2$ where h is the hypotenuse of the triangle. The work is identical to the above but with a different denominator. It all carries through, and gives us half of the above answer, i.e. $\frac{324}{5} = 64.8$.

§1.2 54. A little thinking, maybe drawing a picture or two, I hope reveals - this is two pieces one each that is a "cap" of a sphere (as pictured in 55). h is the distance to the bottom of the cap and r is the distance to the top of the cap. The cross-section slices are circles. The radii of the circles, for any given height y can be found from the Pythagorean theorem (which I call the multi-cultural right triangle theorem, ask me why if you want) to be $\sqrt{r^2-y^2}$. So, the two caps together (hence the 2 in front) $= 2\int_h^r \pi(r^2-y^2)dy$ (notice that the square and square root nicely canceled this time). $= 2\pi(r^2y-\frac{1}{3}y^3)|_h^r = 2\pi[(r^3-\frac{1}{3}r^3)-(r^2h-\frac{1}{3}h^3)] = \frac{2\pi}{3}(2r^3-3hr^2+h^3)$. This happens to factor, though I'm not sure it's useful $= \frac{2\pi}{3}(r-h)^2(2h+h)$. I do *not* see how it relates to the answer in the book. I will give +2 on this question for the first person to write me an email correctly explaining the book's answer *or* explaining the mistake they made to get that answer.

EXTRA BONUS! §1.2 8. Find the volume of a tetrahedron with base side of 4 units. Ok, we started this question in class, and what I did was mostly fine, but there's a tricky step at the end of setup. But the beginning was good: for an equilateral triangle of sidelength a, the altitude is $\frac{\sqrt{3}}{2}a$. That just came from splitting the triangle in half, and the altitude is the same as a perpendicular bisector for any equilateral (in fact isosceles) triangle. From this, the area of any equilateral triangle is given by $\frac{1}{2}a(\frac{\sqrt{3}}{2}a) = \frac{\sqrt{3}}{4}a^2$. We'll need this. The one other thing we need is the height of the tetrahedron. And that's where I stalled in class (or was wrong, I can admit it, I surely said some not quite correct things there). To get there, I like looking at a particular triangle in the tetrahedron.



One side is a side of the tetrahedron and the other vertex is at the midpoint of the opposite side. This triangle is perpendicular to the base of the tetrahedron and it contains the vertex. Therefore the altitude of this triangle is the height of the tetrahedron, which is what we want. One side of this triangle is 4. The other two sides are medians of the equilateral triangle so they have length $\frac{\sqrt{3}}{2}4$. My idea for this triangle is to find it's area in two ways, one way using the 4 side as the base, and one way using the bottom side as the base. The first way is not so bad, because the 4 side is the base of an isosceles triangle, so again we can split it in two, giving a right triangle with one side $\frac{4}{2}$, and one side $\frac{\sqrt{3}}{2}4$. The multicultural right triangle theorem tells us that the other side is $\frac{4}{\sqrt{2}}$. Therefore the area of the triangle is $\frac{1}{2}4(\frac{4}{\sqrt{2}})$. This is a set-up. Using the bottom of the triangle as the base, the area is *also* $\frac{1}{2}\frac{\sqrt{3}}{2}4h$ where h is the desired height of the tetrahedron. So, we can set these two areas equal $\frac{1}{2}4(\frac{4}{\sqrt{2}}) = \frac{1}{2}\frac{\sqrt{3}}{2}4h$ and solve for h to get the intriguing-looking $\sqrt{\frac{2}{3}}4$ for the height of our tetrahedron. That should be the key we need. Next we need the scaling. to get our side-lengths to line up. At $y = 0$ we need a side length of 0 (again we're working point down to be easier), and at $y = \sqrt{\frac{2}{3}}4$ we need a side length of 4. Apparently we must multiply y by $\sqrt{\frac{3}{2}}$. So, we can now integrate the cross-section equilateral triangles, as given above, from 0 to the height: $\int_0^{\sqrt{\frac{2}{3}}4} \frac{\sqrt{3}}{4}(\sqrt{\frac{3}{2}}y)^2 dy = \frac{3\sqrt{3}}{8}\int_0^{\sqrt{\frac{2}{3}}4} y^2 dy = \frac{\sqrt{3}}{8}y^3|_0^{\sqrt{\frac{2}{3}}4} = \frac{\sqrt{3}}{8}(\sqrt{\frac{2}{3}}4)^3 = \frac{\sqrt{2}}{12}4^3$. And, you may ask ... why did I never compute with 4? Because I wanted the general formula, so if a tetrahedron has a side length of a, it's volume is given by $\frac{\sqrt{2}}{12}a^3$.