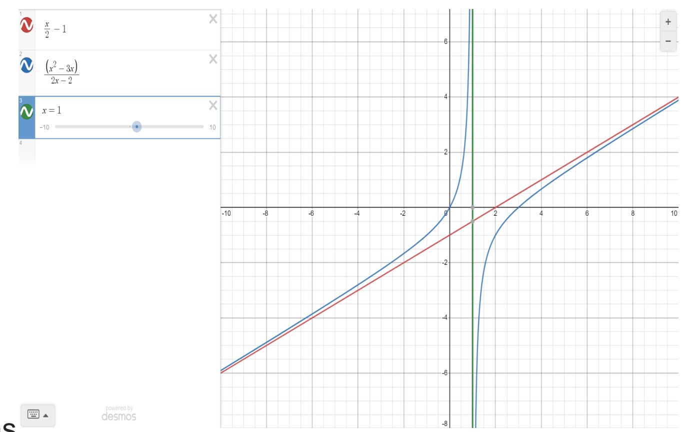
Kevin Rogers Test Objective 8

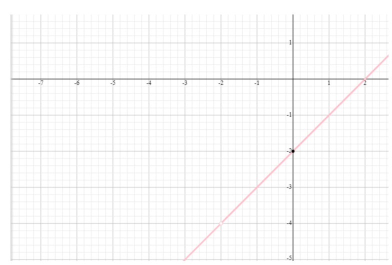
0008 - Understand the Properties of Rational, Radical, and Absolute Value Functions and Relations

Exemplar 1: Analyzing the properties of a given function.
Domain: The set of all possible elements that the independent variable can have. Generally, the domain is defined as all possible values of x.
Range: The set of all possible values of the dependent variable that are obtained by using all the values of the domain as input. We can consider the range as all possible values of y.
Two ways to display Domain and Range: 1) Set Notation: $\{x|x\in R\}$ 2) Interval Notation: $(-\infty, \infty)$
Horizontal Asymptote: As x approaches infinity or negative infinity, the function approaches some value b.

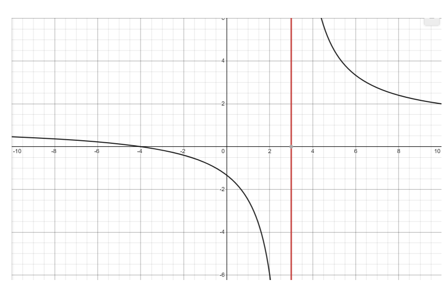


Vertical Asymptote: As x approaches some constant value from either the left or the right, then the curve goes toward infinity or negative infinity.
Oblique Asymptote: As x approaches infinity or negative infinity, the function goes toward a line y=mx+b.
Example: $f(x)=(x^2-3x)/(2x-2)$
Vertical Asymptote: x=1
Oblique Asymptote: y=(x/2)-1
*It is important to note that Rational, Radical, and absolute value functions are NOT linear. Linear graphs create a straight line throughout the graph and none of these functions are straight throughout.

Exemplar 2: Solving Problems involving rational, radical, and absolute value functions using various algebra techniques.
Rational function: A quotient of two polynomials in the form f(x)=p(x)/q(x).
Types of discontinuities with rational functions:
Point Discontinuity: If the right hand limit is equal to the left hand limit at a but not equal to f(a). This can be found algebraically when factors from the numerator and denominator cancel.



Ex: $f(x)=(x^2-4)/(x+2)$
There is a point discontinuity at x=-2 since (x+2) Cancels in both the numerator and denominator.
Infinity Discontinuity: If the right hand limit and left hand limit as x approaches a is infinity or negative infinity. This can be found when after canceling factors in the numerator and denominator making q(x)=0 and create vertical asymptotes.



Ex: $f(x)=(x^3-5)/(x-3)$
Since (x-3) does not cancel with anything in the numerator, there is an infinite discontinuity at x=3.

Tips for finding discontinuities and asymptotes:
i) The excluded values of the domain of a rational function help to identify the vertical asymptotes.
ii) The excluded values of the range of a rational function help to identify the Horizontal asymptote (note: there can only be a maximum of one per function)
iii) The linear factors that get canceled when a rational function is simplified would give us the point discontinuities.

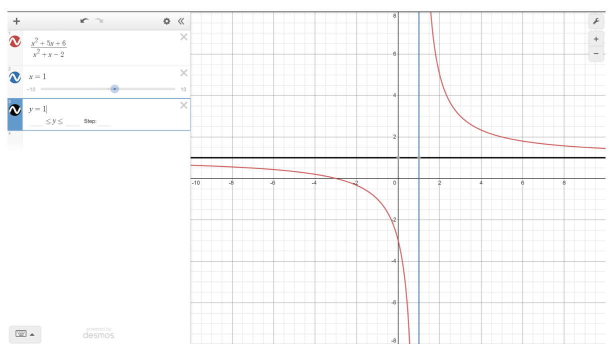
Finding Oblique Asymptotes: In the rational function, the degree of $p(x)$ must be 1 greater than the degree of $q(x)$. Then perform polynomial long division of $p(x)/q(x)$. The dividend in the form $y=mx+b$ is the oblique asymptote.
Ex: $f(x)=(x^3+2x^2-5x+10)/(x^2-5x+4)=x+7+(26x-18)/x^2-5x+4$
Therefore, the Oblique Asymptote is $y=x+7$

<u>Radical function</u>: A function where the variable occurs in the radicand, in the form $f(x)=a\sqrt[n]{g(x)}+k$, where n is an integer.
There are cases where you have to reject answers to radical functions since it is not in the domain. You must test it with the original function to see if it is in the domain.
Note: $\sqrt{(x^2+y^2)}\neq x+y$ and $(1/x)+(1/y)\neq 1/(xy)$

<u>Absolute value function</u>: A function of the form $f(x)=a|x-h|+k$
Absolute value is defined as the distance between a number and 0 without taking into consideration direction.
Absolute value is also defined as $f(x)=|x|$ and
$f(x)=\{x$, when $x\geq 0$
$-x$, when $x<0\}$
Property of Non-negativity: $|a|\geq 0$
Property of Subadditivity: $|a+b|\leq|a|+|b|$
Proof:
Case 1: $a\geq 0$ and $b\geq 0$
Both sides would equal a+b since a and b are both non-negative.
For both cases 2 and 3 we do not lose generality and the same proof can be used in switching a and b.
Case 2: $a\geq 0$ and $b<0$ where $|a|>|b|$
$a+b<a-b \Rightarrow b<-b$ since b is negative
Case 3: $a\geq 0$ and $b<0$
$-a-b<a-b \Rightarrow -a<a$
Case 4: $a<0$ and $b<0$
Both sides will equal $-(a+b)$ since a and b are both negative.
Therefore, $|a+b|\leq|a|+|b|$. When a and b are the same sign they are equal. When a and b have different signs, $|a+b|<|a|+|b|$.
Property of Symmetry: $|-a|=|a|$
Property of Multiplicatively: $|a|*|b|=|a*b|$

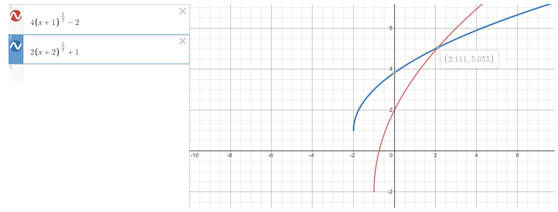
Property of Triangle Inequality:|a-b|$\leq$|a-b|+|c-a| (Can be proved by the Property of subadditivity)
Exemplar 3: Modeling and Solving problems graphically using systems of equations and inequalities involving rational, radical, and absolute value functions.
<u>Rational function</u>: A quotient of two polynomials in the form f(x)=p(x)/q(x).
The *domain* of rational functions are all values of x such that q(x)≠0
The *range* of a rational function are all values in the domain of the inverse function.
The *horizontal asymptote* of a rational function depends on the degree of the numerator and denominator and there are three cases.
If i) The degree of the numerator is greater than the degree of the denominator then there is no horizontal asymptote.
ii) The degree of the numerator is equal to the degree of the denominator then the horizontal asymptote is y=(leading coefficient of numerator)/(leading coefficient of denominator).
iii) The degree of the numerator is less than the degree of the denominator then the horizontal asymptote is y=0.
The *vertical asymptote* of a rational function is found by simplifying the function by canceling any common factors in the numerator and denominator and then setting the denominator equal to 0. If you have an *odd* multiplicity at a vertical asymptote, one end approaches infinity and the other approaches negative infinity. If you have an even multiplicity at a vertical asymptote, both ends approach either infinity or negative infinity
The *oblique asymptote* of a rational function is only when the degree of the numerator is one greater than the degree of the denominator. It is y = the quotient obtained by dividing the numerator by the denominator using long division.

Ex: f(x)=(x^2+5x+6)/(x^2+x-2)
Domain: 0=x^2+x-2 => x=-2Vx=1
Range: f^{-1}(x)=(3+x)/(x-1)
0≠x-1 => x≠1 => (-∞, 1),(1, ∞)
Horizontal Asymptote: y=1/1=1
Vertical Asymptote:
f(x)=(x+2)(x+3)/((x+2)(x-1))
f(x)=(x+3)/(x-1) Odd multiplicity
0=x-1 => x=1



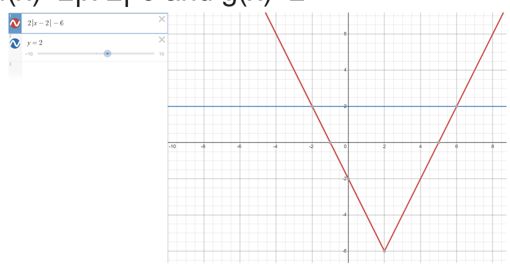
<u>Radical function</u>: A function where the variable occurs in the radicand, in the form f(x)=a $\sqrt[n]{g(x)}$+k, where n is an integer.
If n is even:
The *domain* is whenever g(x)$\geq$0 and the domain of g(x).
The *range* if a is positive is [k, ∞). The range if a is negative is (-∞, k].
If n is odd:
The *domain* is the domain of g(x).
The *range* is (-∞, ∞).
Ex: Find the point of intersection between f(x)=4√(x+1)-2 and g(x)=2√(x+2)+1
f(x)=g(x)
Plug both functions into your graphing calculator
2nd->Calc->Intersect

Click on left, right, and guess of the point



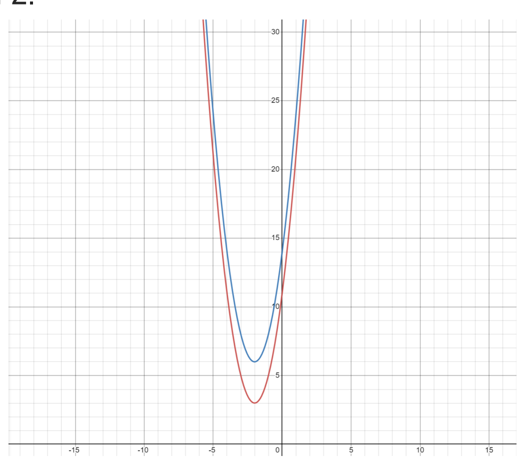
<u>Absolute value function</u>: A function of the form $f(x)=a|x-h| +k$
If a is positive, the function opens upward. If a is negative, the function opens downward.
The *domain* is the domain of $(-\infty, \infty)$
The *range* if a is positive if $[k, \infty)$. The range if a is negative is $(-\infty, k]$.

Ex: Find the area of the triangle created by $f(x)=2|x-2|-6$ and $g(x)=2$
$2|x-2|-6<2$
$A=½*b*h=½*8*8=32$ square units.

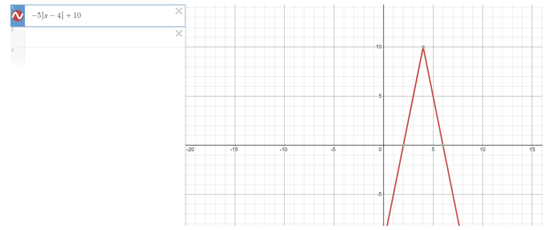


Exemplar 4: Interpreting and analyzing the effects of transformations on the graph of a function.
Function in the form $f(x)=a(x-h)^n+k$ where $n > 0$ (Note: n could equal a fraction, such as ½, or $f(x)=a\sqrt{(x-h)}+k$) or absolute value functions in the form $f(x)=a|x-h| +k$
If $|a|>1$, then the function vertically stretches by a factor of a.
If $|a|<1$, then the function is vertically compressed by a factor of a.
If $a<0$, then the function reflects over the x axis.
If $a<0$, then the function does not reflect over the x axis.
If $h>0$, then the function moves right h units.
If $h<0$, then the function moves left -h units.
If $k>0$, then the function moves up k units.
If $k<0$, then the function moves down -k units.

Why do we do stretches and compressions before transformations?
Graph $f(x)=2(x+2)^2+3$ first by doing the transformation 2 units left and 3 units up. Then vertically stretch the function in respect to the x-axis by a factor of 2.
Find f(2) both algebraically and graphically.
f(2)=35 (red) but on the graph it says f(2)=38 (blue)
Therefore, we can see that we need to do vertical Stretches and compressions before we do translations. This rule also applies to reflections. You must do the reflection first before you do any translation.



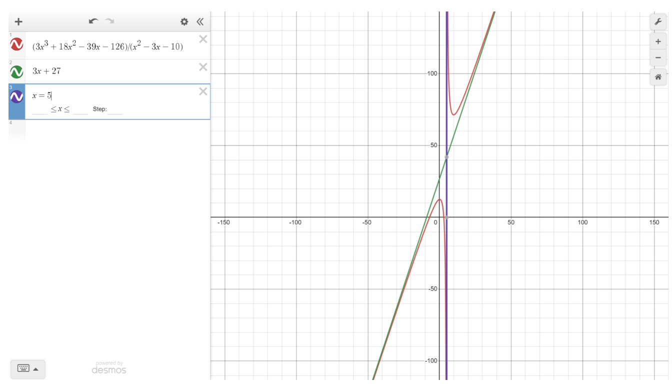
Exemplar 5: Applying the properties of rational, radical, and absolute value functions and relations to model and solve problems.



What is the maximum of the function f(x)=-5|x-4|+10?
We know that if the range is negative, then the function opens downward. Since a=-5, we know the range is (-∞,10]. Therefore, the maximum value of f(x) is 10.

Solving absolute value equations: 1) Isolate the absolute value
2) Create the two possible equations by making the absolute values positive and negative (Ex: x-h=f(x) and -(x-h)=f(x))
3) Solve for both values of x
Ex: 4|x-3|+6=36 => 4|x-3|=28 => |x-3|=7 => x-3=7 V -(x-3)=7 => x=10 V x=-4

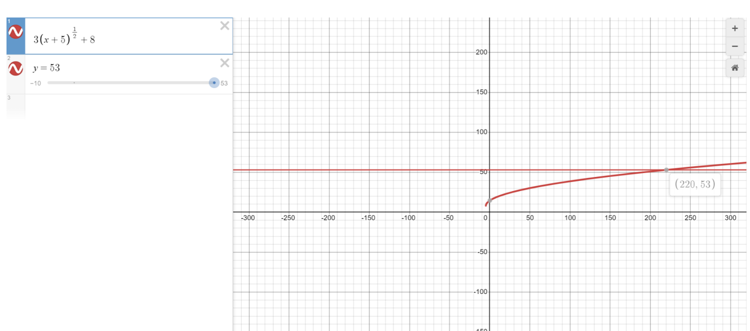
Find all asymptotes and discontinuities in the function $f(x)=(3x^3+18x^2-39x-126)/(x^2-3x-10)$
f(x)=(3(x-3)(x+2)(x+7))/((x+2)(x-5)). Since (x+2) cancels in both the numerator and denominator, we know there is a point discontinuity at x=-2.
We then have 0=x-5 to find the vertical asymptotes, so x=5 is the vertical asymptote. We know that one side of x=5 approaches infinity while the other side approaches negative infinity. There is an infinity discontinuity.
Also, since the degree of the numerator is one greater than the denominator, we are going to have an oblique asymptote.
$(3x^3+18x^2-39x-126)/(x^2-3x-10)=3x+27+72/(x-5)$.
Therefore, we have an oblique asymptote y=3x+27.
Point discontinuity at x=2



Vertical asymptote: x=5
Oblique Asymptote: y=3x+27

Solving radical equations: 1) Isolate the radical to one side
2) Raise both sides to the nth power of the root
3) Solve the remaining equation

Solve the equation 53=3√(x+5)+8.
Domain: [-5,∞)
53=3√(x+5)+8 => 45=3√(x+5) => 15=√(x+5) => 225=x+5 => 220=x
Therefore, f(220)=53.



Questions:

1. What is the vertical asymptote of the function $f(x)=(x^2-3x-10)/(x-6)$? Do both sides of the asymptote approach the same infinity? Explain

2. What is the oblique asymptote of the function $f(x)=(x^2-3x-10)/(x-6)$?

3. What is the horizontal asymptote in a function where the numerator is degree 2 and the denominator is degree 3?

4. List the discontinuities in the function $f(x)=((x+3)(x-4)(x+5))/((x+1)(x-4))$.

5. State the domain of the functions $f(x)=3\sqrt{(2x-8)}+5$.

6. Solve the equation $3x+5=\sqrt{(2x+3)}-4$.

7. Solve the equation $15=4|x+8|-1$.

8. Find the area in the inequality $y<8$, $y>3$, and $y=-|x-5|+10$.

9. Prove $\sqrt{(x^2+y^2)} \neq x+y$ and $(1/x)+(1/y) \neq 1/(xy)$ (hint: use counterexamples).

10. Graph $f(x)=2(x+7)^2+5$.

Questions Key

1. What is the vertical asymptote of the function $f(x)=(x^2-3x-10)/(x-6)$? Do both sides of the asymptote approach the same infinity? Explain.

$0=x-6 \Rightarrow x=6$.
There is a vertical asymptote at $x=6$. Both sides of the asymptote approach different infinities since there is a multiplicity of one in the denominator.

2. What is the oblique asymptote of the function $f(x)=(x^2-3x-10)/(x-6)$?

$(x^2-3x-10)/(x-6)=x+3+8/(x-6)$. Therefore, there is an oblique asymptote $y=x+3$.

3. What is the horizontal asymptote in a function where the numerator is degree 2 and the denominator is degree 3?

The horizontal asymptote of a rational function in which the degree of the denominator is greater than the degree of the numerator is $y=0$.

4. List the discontinuities in the function $f(x)=((x+3)(x-4)(x+5))/((x+1)(x-4))$.

Since $(x-4)$ cancels in the numerator and denominator, we have a point discontinuity at $x=4$
$x+1=0 \Rightarrow x=-1$. Therefore, there is an infinite discontinuity at $x=-1$.

5. State the domain of the functions $f(x)=3\sqrt{(2x-8)}+5$.

$2x-8\geq 0 \Rightarrow 2x\geq 8 \Rightarrow x\geq 4$

6. Solve the equation $3x+5=\sqrt{(2x+81)}-4$.

$3x+5=\sqrt{(2x+81)}-4 \Rightarrow 3x+9=\sqrt{(2x+81)} \Rightarrow (3x+9)^2=2x+81 \Rightarrow 9x^2+54x+81=2x+81$
$\Rightarrow 9x^2+52x=0 \Rightarrow x(9x+52)=0 \Rightarrow x=0 \vee x=-52/9$

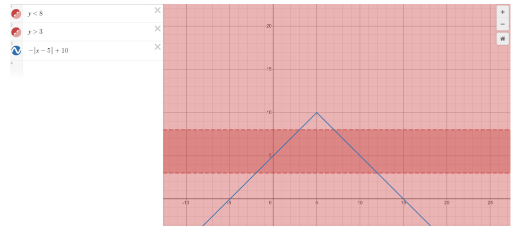
7. Solve the equation $15=4|x+8|-1$.

$15=4|x+8|-1 \Rightarrow 16=4|x+8| \Rightarrow 4=|x+8| \Rightarrow 4=x+8 \vee 4=-(x+8) \Rightarrow -4=x \vee -12=x$

8. Find the area in the inequality $y<8$, $y>3$, and $y=-|x-5|+10$.

Domain: (-2,12)

$A=\frac{1}{2}*b*h=\frac{1}{2}*8*5=20$ units squared.

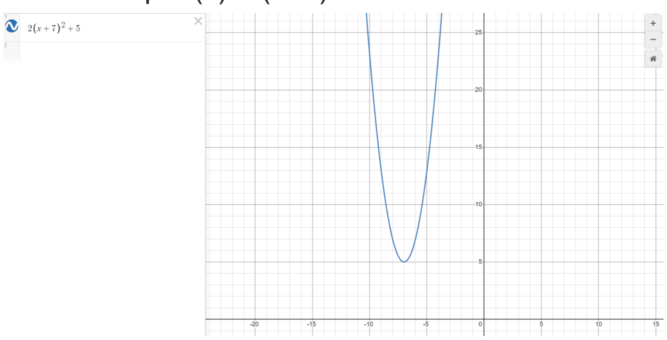


9. Prove $\sqrt{(x^2+y^2)}\neq x+y$ and $(1/x)+(1/y)\neq 1/(xy)$ (hint: use counterexamples).

$\sqrt{(x^2+y^2)}=\sqrt{((2)^2+(1)^2)}=\sqrt{(4+1)}=\sqrt{5}$ $x+y=2+1=3$ $\sqrt{5}\neq 3$

$(1/x)+(1/y)=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ $1/(xy)=1/((2)(3))=\frac{1}{6}$ $\frac{5}{6}\neq\frac{1}{6}$

10. Graph $f(x)=2(x+7)^2+5$.



Sources:

https://www.mathsisfun.com/algebra/finding-intercepts-equation.html

https://www.mechamath.com/algebra/domain-and-range-of-linear-functions/

https://www.desmos.com/calculator

https://www.cuemath.com/calculus/rational-function/

https://www.cuemath.com/algebra/absolute-value-function/

https://www.vedantu.com/maths/absolute-values

https://www.alamo.edu/contentassets/ab5b70d70f264cec9097745e8f30ca0a/functions/math0303-translation-of-functions.pdf

Solving Radical Equations | College Algebra (lumenlearning.com)