This example illustrates what formal proofs that a function is an injection, surjection, or bijection might look like. All the proofs concern the function

$$f(x, y) = (x + y, x - y) \tag{1}$$

We begin by showing that f is an injection and a surjection.

Theorem 1. *The function f defined by (1) is an injection.*

Proof. We will prove that f is an injection by showing that $f(a, b) = f(c, d)$ implies that $a = b$ and $c = d$. From the definition of f, $f(a, b) = f(c, d)$ implies the following system of equations.

$$\begin{aligned} a + b &= c + d \\ a - b &= c - d \end{aligned}$$

Adding these equations yields

$$2a = 2c$$

from which we conclude that $a = c$. Substituting this back into the first equation then produces

$$a + b = a + d$$

indicating that $b = d$. We have now shown that if $f(a, b) = f(c, d)$ it must be true that $a = b$ and $c = d$, which in turn establishes that f is an injection. □

Theorem 2. *Function f defined by (1) is a surjection.*

Proof. We show that f is a surjection by showing that for any pair of real numbers, (x, y), there is another pair (a, b) such that $f(a, b) = (x, y)$. From the definition of f, this would require

$$\begin{aligned} a + b &= x \\ a - b &= y \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} a &= x - b \\ a &= y + b \end{aligned}$$

so that $x - b = y + b$. We can then isolate b on one side of this equation:

$$\begin{aligned} x - b &= y + b \\ x - y &= 2b \\ b &= \frac{x - y}{2} \end{aligned}$$

Now we plug this definition of b back into one of the equations for a to get

$$\begin{aligned} a &= x - b \\ &= x - \frac{x-y}{2} \\ &= \frac{2x - x + y}{2} \\ &= \frac{x+y}{2} \end{aligned}$$

We finish the proof by verifying that for any pair of real numbers (x, y), $f(\frac{x+y}{2}, \frac{x-y}{2}) = (x, y)$:

$$\begin{aligned} f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) &= \left(\frac{x+y}{2} + \frac{x-y}{2}, \frac{x+y}{2} - \frac{x-y}{2}\right) \\ &= \left(\frac{x+y+x-y}{2}, \frac{x+y-x+y}{2}\right) \\ &= \left(\frac{2x}{2}, \frac{2y}{2}\right) \\ &= (x, y) \end{aligned}$$

We have now shown that every pair (x, y) in the codomain of function f has a preimage under f, and so f is a surjection. □

Finally, being an injection and a surjection implies that f is a bijection:

Corollary 1. *Function f defined by (1) is a bijection.*

Proof. That f is a bijection follows immediately from it being an injection and a surjection. □