This document presents some examples of proofs by contradiction. The first is a generic demonstration of the technique:

Theorem 1. *For all integers n, if $n \equiv 1 \pmod 2$ then $n \not\equiv 2 \pmod 4$.*

Proof. We will prove by contradiction that for all integers n, if $n \equiv 1 \pmod 2$ then $n \not\equiv 2 \pmod 4$. We assume for the sake of contradiction that n is an integer such that $n \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$. Since $n \equiv 1 \pmod 2$, there is some integer a such that

$$\begin{aligned} n - 1 &= 2a \\ n &= 2a + 1 \end{aligned} \tag{1}$$

Similarly, since $n \equiv 2 \pmod 4$, there is some other integer b such that

$$\begin{aligned} n &= 4b + 2 \\ &= 2(2b + 1) \end{aligned} \tag{2}$$

From equation (1), we see that n is odd, and from equation (2) that n is even. Since a number can't be both odd and even, this is a contradiction. Since the assumption that $n \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$ leads to a contradiction, we have proven by contradiction that for all integers n, if $n \equiv 1 \pmod 2$ then $n \not\equiv 2 \pmod 4$. □

The next example is based on a classic proof by contradiction, namely a proof that $\sqrt{2}$ is irrational. Our example adapts the ideas from that proof to proving that $\sqrt{6}$ is also irrational:

Theorem 2. *If x is a real number such that $x^2 = 6$ then x is irrational*

Proof. We assume that x is a real number and $x^2 = 6$, and will prove by contradiction that x is irrational. So assume for the sake of contradiction that not only is x a real number and $x^2 = 6$, but that x is rational. Since x is rational, it can be written as

$$x = \frac{m}{n} \tag{3}$$

where m and n are integers, $n \neq 0$, and m and n have no common factors besides 1.

Squaring both sides of (3) gives

$$\begin{aligned} 6 &= \frac{m^2}{n^2} \\ 6n^2 &= m^2 \end{aligned} \tag{4}$$

or

$$m^2 = 2(3n^2)$$

which means that m must be even. In other words $m = 2p$ for some integer p.

Plugging $2p$ into equation (4) in place of m gives

$$\begin{aligned}6n^2 &= (2p)^2 \\ &= 4p^2\end{aligned}$$

or

$$3n^2 = 2p^2$$

meaning that $3n^2$ is even. Now, if n^2 were odd, $3n^2$ would also be odd, so n^2 must be even.

We now arrive at a contradiction, because if m and n are both even, they have a common factor of 2, but we defined them to have no such common factor. Since we have showed that the assumption that x is rational leads to a contradiction, any square root of 6 must in fact be irrational. □