Math 239 01
Prof. Doug Baldwin

Problem Set 9 — Functions

Complete by **Monday, April 22**
Grade by **Wednesday, April 24**

Purpose

This problem set further develops your ability to work with and reason about functions. In particular, when you finish this problem set you should be able to ...

- Prove that functions are injections, surjections, and/or bijections
- Prove theorems about function compositions
- Prove theorems about inverse functions
- Write formal proofs.

Background

This exercise draws on material from sections 6.1 through 6.5 of our textbook. We discussed that material in classes between April 10 and 15.

Activity

Solve the following problems. All proofs must be written according to conventions for formal proofs, including typeface rules (e.g., italic variables, emphasized labels for theorems and proofs, etc.).

Question 1. (Exercise 8a in section 6.4 of our textbook.)

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$. Find an expression for $f^n(x)$ (informally, the notation f^n means the n-fold composition of f with itself; see the textbook for a precise formal definition, and ask questions if you have any). Use induction to prove your expression correct.

Solution:

Theorem 1. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x+1$, then for all natural numbers n, $f^n(x) = x+n$.

Proof. We use induction on n to prove that for all natural numbers n, $f^n(x) = x + n$.

For the basis step, note that $f^1(x) = f(x) = x + 1 = x + n$. Thus the theorem holds when $n = 1$.

For the induction step, we assume that $f^k(x) = x + k$, and show that $f^{k+1}(x) = x + (k+1)$. From the formal definition of f^n, we have

$$\begin{aligned} f^{k+1}(x) &= (f \circ f^k)(x) \\ &= f(f^k(x)) \\ &= f(x+k) \\ &= (x+k)+1 \\ &= x+(k+1) \end{aligned}$$

We have thus shown that $f^1(x) = x + 1$ and that if $f^k(x) = x + k$ then $f^{k+1}(x) = x + (k+1)$, and so have proven by induction that for all natural numbers n, $f^n(x) = x + n$. □

Question 2. (Exercise 9 in section 6.5 of our textbook.)

Prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Solution:

Theorem 2. If function $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Proof. We assume that function $f : A \to B$ is a bijection, and show that $f^{-1} : B \to A$ is also a bijection. We first note that by Theorem 6.26, since f is a bijection, f^{-1} is a function and for all $a \in A$ and all $b \in B$, $f(a) = b$ if and only if $f^{-1}(b) = a$. In particular, this means that for all pairs $(b, a) \in f^{-1}$, the pair $(a, b) \in f$, so f is the inverse of f^{-1}. We know that f and f^{-1} are both functions, and by Theorem 6.25, the inverse of a function is a function if and only if the original function is a bijection. Thus, since the inverse of f^{-1} is a function, f^{-1} must be a bijection. □

Question 3. (An extension of exercise 14 in section 6.3 of our textbook.)

Determine whether function $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4} \tag{1}$$

is an injection, a surjection, and/or a bijection. Justify each conclusion with a formal proof or a counterexample. See the book for hints and more details.

I expect that answering this question with complete formal proofs will be time-consuming, and that the proofs will be long, will involve multiple proof techniques, and may benefit from being written as multiple distinct proofs. Please take the time to think about these things as you write your answer to this question, possibly including taking the time to revise your first version of an answer into one you feel is clearer or simpler. These are all things mathematicians go through in devising "real" (as opposed to classroom) proofs.

Solution: It is convenient to begin with a lemma that helps in showing both that f is an injection and that it is a surjection.

Lemma 1. If function $f : \mathbb{N} \to \mathbb{Z}$ is defined by Equation (1), then for all natural numbers n, $f(n) > 0$ if and only if n is even.

Proof. We prove that $f(n) > 0$ if and only if n is even by proving each direction separately.

For the "if" direction, we assume that n is even, and show that $f(n) > 0$. Since n is even, $(-1)^n$ is positive. Thus $f(n)$ becomes

$$\begin{aligned} f(n) &= \frac{1+(-1)^n(2n-1)}{4} \\ &= \frac{1+(2n-1)}{4} \\ &= \frac{2n}{4} \\ &= \frac{n}{2} \end{aligned}$$

Since n is a natural number, $n \geq 1$, and so $\frac{n}{2} > 0$.

For the "only if" direction, we show that if $f(n) > 0$ then n is even by proving the contrapositive. In other words, we assume that n is odd, and show that $f(n) \leq 0$. Since n is odd, $(-1)^n$ is negative, and so $f(n)$ becomes

$$\begin{aligned} f(n) &= \frac{1+(-1)^n(2n-1)}{4} \\ &= \frac{1-(2n-1)}{4} \\ &= \frac{2-2n}{4} \\ &= \frac{1-n}{2} \end{aligned}$$

Since n is an odd natural number, $n \geq 1$, and so $1-n \leq 0$, and therefore $\frac{n-1}{2} \leq 0$ also.

Since we have now shown that if n is even, $f(n) > 0$, and that if $f(n) > 0$, n is even, we have proven that $f(n) > 0$ if and only if n is even. □

We now use this lemma to prove that f is an injection.

Theorem 3. The function $f : \mathbb{N} \to \mathbb{Z}$ defined in Equation (1) is an injection.

Proof. We prove that f is an injection by showing that if $f(n) = f(m)$, then $n = m$. Thus, assume that n and m are natural numbers such that $f(n) = f(m)$, in other words

$$\frac{1+(-1)^n(2n-1)}{4} = \frac{1+(-1)^m(2m-1)}{4}$$

By multiplying both sides of this equation by 4, we see that

$$1+(-1)^n(2n-1) = 1+(-1)^m(2m-1) \tag{2}$$

We now consider two cases, namely n is odd and n is even.

For the first case, suppose n is odd. Then by Lemma **??**, $f(n) \leq 0$. Since $f(n) = f(m)$, $f(m)$ must also be less than or equal to 0, and so again by Lemma **??**, m is also odd. Thus Equation (**??**) becomes

$$1-(2n-1) = 1-(2m-1)$$

which, by canceling like terms on each side, implies that $n = m$.

For the second case, suppose n is even. Then, by an argument similar to that in the first case, $f(n)$ is positive and thus so is $f(m)$, and m is also even. Then Equation (**??**) becomes

$$1 + (2n - 1) = 1 + (2m - 1)$$

Again as in the first case, canceling like terms on both sides of this equation yields $n = m$.

By showing that $f(n) = f(m)$ implies that $n = m$ both when n is odd and when n is even, we have shown that for all natural numbers n and m, i.e., all n and m in the domain of f, that $f(n) = f(m)$ implies $n = m$. This in turn establishes that the function $f : \mathbb{N} \to \mathbb{Z}$ defined in Equation (1) is an injection. □

We next turn to showing that f is a surjection.

Theorem 4. The function $f : \mathbb{N} \to \mathbb{Z}$ defined in Equation (1) is a surjection.

Proof. We show that the function $f : \mathbb{N} \to \mathbb{Z}$ defined in Equation (1) is a surjection by showing that every element of its codomain, $\mathbb{Z}$, is the image of some element of the domain. Let y be an integer. We show that y has a preimage under f in two cases, corresponding to $y \leq 0$ and $y > 0$.

In the first case, suppose $y \leq 0$. Then by Lemma **??**, the preimage of y, if it exists at all, must be odd, and so

$$\begin{aligned} y &= f(n) \\ &= \frac{1 - (2n - 1)}{4} \\ &= \frac{2 - 2n}{4} \\ &= \frac{1 - n}{2} \end{aligned}$$

Solving for n in terms of y, we see that

$$\begin{aligned} y &= \frac{1 - n}{2} \\ 2y &= 1 - n \\ 2y - 1 &= -n \\ n &= 1 - 2y \end{aligned}$$

Since y is an integer less than or equal to 0, $1 - 2y$ is a natural number and so is in the domain of f. We verify that $f(1 - 2y)$ is indeed equal to y as follows:

$$f(1 - 2y) = \frac{1 + (-1)^{1-2y}(2(1 - 2y) - 1)}{4}$$

Now $2y$ is an even integer less than or equal to 0, so $1 - 2y$ is odd, and this equation simplifies to.

$$\begin{aligned} f(1 - 2y) &= \frac{1 - (2(1 - 2y) - 1)}{4} \\ &= \frac{1 - (2 - 4y - 1)}{4} \\ &= \frac{1 - 2 + 4y + 1}{4} \\ &= \frac{4y}{4} \\ &= y \end{aligned}$$

In the second case, suppose $y > 0$. Then the preimage of y must be even, and so

$$\begin{aligned} y &= f(n) \\ &= \frac{1 + (2n - 1)}{4} \\ &= \frac{2n}{4} \\ &= \frac{n}{2} \end{aligned}$$

This implies that $n = 2y$. Since y is an integer greater than 0, $2y$ is a natural number. We can verify that $f(2y) = y$ as follows:

$$\begin{aligned} f(2y) &= \frac{1 + (-1)^{2y}(2(2y) - 1)}{4} \\ &= \frac{1 + (2(2y) - 1)}{4} \\ &= \frac{1 + 4y - 1}{4} \\ &= \frac{4y}{4} \\ &= y \end{aligned}$$

We have now proven that y has a preimage under f both when $y \leq 0$, and when $y > 0$. Thus all integers are the image under f of some natural number, and so f is a surjection. □

Finally, the previous two theorems imply that f is a bijection:

Corollary 1. The function $f : \mathbb{N} \to \mathbb{Z}$ defined in Equation (1) is a surjection.

Proof. Function f is an injection and a surjection, and so is a bijection. □

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.