Math 239 01
Prof. Doug Baldwin

Problem Set 8 — Sets

Complete by **Sunday, April 14**
Grade by **Tuesday, April 16**

Purpose

This problem set further develops your ability to work with and reason about sets. In particular, when you finish this problem set you should be able to ...

- Prove theorems about sets using set algebra
- Calculate Cartesian products
- Prove theorems about Cartesian products
- Carry out calculations involving indexed families of sets
- Prove theorems about indexed families of sets
- Write formal proofs.

Background

This exercise draws on material from sections 5.3 through 5.5 of our textbook. We discussed that material in class on March 29 (section 5.3, set algebra)), April 5 (section 5.4, Cartesian product) and April 8 (section 5.5, indexed families of sets).

Activity

Solve the following problems. All proofs must be written according to conventions for formal proofs, including typeface rules (e.g., italic variables, emphasized labels for theorems and proofs, etc.).

Question 1. (Based on exercise 5 in section 5.3 of our textbook.)

Use Venn diagrams to form a conjecture about the relationship between $A-(B\cap C)$ and $(A-B)\cup(A-C)$, where A, B, and C are subsets of some universal set. Then use set algebra to prove your conjecture.

Solution:

Conjecture 1. If A, B, and C are subsets of some universal set, then $A-(B\cap C) = (A-B)\cup(A-C)$.

Proof. We prove that if A, B, and C are subsets of some universal set, then $A - (B \cap C) = (A - B) \cup (A - C)$ by algebraically transforming the left-hand side into the right-hand side. Specifically,

$$\begin{aligned} A - (B \cap C) &= A \cap (B \cap C)^C \\ &= A \cap (B^C \cup C^C) \\ &= (A \cap B^C) \cup (A \cap C^C) \\ &= (A - B) \cup (A - C) \end{aligned}$$

Thus, whenever A, B, and C are subsets of some universal set, $A - (B \cap C) = (A - B) \cup (A - C)$. □

Question 2. Let $A = \{-1, 0, 1\}$ and $B = \{a, b\}$. Find the Cartesian products. . .

1. $A \times B$
2. $B \times A$
3. $\emptyset \times A$

Solution:

1. $A \times B = \{(-1, a), (0, a), (1, a), (-1, b), (0, b), (1, b)\}$
2. $B \times A = \{(a, -1), (b, -1), (a, 0), (b, 0), (a, 1), (b, 1)\}$
3. $\emptyset \times A = \emptyset$ because there are no ordered pairs (x, y) satisfying the requirement that x be an element of $\emptyset$.

Question 3. Problem 9 from section 5.4 of our textbook (determine whether it is true that if $A \times B = A \times C$ for sets A, B, and C with $A \neq \emptyset$, then $B = C$, proving your answer and explaining where the requirement that $A \neq \emptyset$ enters the argument).

Solution:

Theorem 1. If A, B, and C are sets with $A \neq \emptyset$, and $A \times B = A \times C$, then $B = C$.

Proof. We assume that A, B, and C are sets with $A \neq \emptyset$, and $A \times B = A \times C$, and show that $B = C$. We show this by showing that $B \subseteq C$ and $C \subseteq B$.

To see that $B \subseteq C$, suppose that $y \in B$, and let x be some element of A. Then the pair (x, y) is an element of $A \times B$, and also an element of $A \times C$, since $A \times B = A \times C$. Thus y must be an element of C. Therefore, $B \subseteq C$.

The proof that $C \subseteq B$ is similar, except with $y \in C$ and then $(x, y) \in A \times C = A \times B$.

Since we have established that $B \subseteq C$ and that $C \subseteq B$, we have proven that if A, B, and C are sets with $A \neq \emptyset$, and $A \times B = A \times C$, then $B = C$. □

That $A \neq \emptyset$ is necessary for there to be an element x to construct the ordered pairs from.

Question 4. Problems 2g and 2h in section 5.5 of our textbook: define $A_n = \{k \in \mathbb{N} | k \geq n\}$ for each natural number n, then find

$$\bigcap_{j \in \mathbb{N}} A_j \quad \text{and} \quad \bigcup_{j \in \mathbb{N}} A_j$$

Solution: Since every natural number n is not in A_{n+1},

$$\bigcap_{j\in\mathbb{N}} A_j = \emptyset$$

Since A_1 is the set of all naturals greater than or equal to 1, i.e., the set of all naturals,

$$\bigcup_{j\in\mathbb{N}} A_j = \mathbb{N}$$

Question 5. Problem 4b in section 5.5 of our textbook (prove part 4 — one of De Morgan's laws — of Theorem 5.30).

Solution:

Theorem 2. If Λ is a non-empty indexing set and $\mathscr{A} = \{A_\alpha | \alpha \in \Lambda\}$ is an indexed family of sets, then

$$\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C = \bigcap_{\alpha\in\Lambda} A_\alpha^C \tag{1}$$

Proof. We prove that if Λ is a non-empty indexing set and $\mathscr{A} = \{A_\alpha | \alpha \in \Lambda\}$ is an indexed family of sets, then Equation (**??**) holds by showing that the each side of the equation is a subset of the other.

To show that $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C \subseteq \bigcap_{\alpha\in\Lambda} A_\alpha^C$, suppose x is an element of $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C$. Then x is not in $\bigcup_{\alpha\in\Lambda} A_\alpha$, i.e., $x \notin A_\alpha$ for all $\alpha \in \Lambda$. This in turn means that $x \in A_\alpha^C$ for all $\alpha \in \Lambda$, and so $x \in \bigcap_{\alpha\in\Lambda} A_\alpha^C$.

To show that $\bigcap_{\alpha\in\Lambda} A_\alpha^C \subseteq \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C$, suppose x is in $\bigcap_{\alpha\in\Lambda} A_\alpha^C$. Then x is in A_α^C for all $\alpha \in \Lambda$, i.e., x is not in any of the A_α. This in turn means that $x \notin \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$, and so x is a member of $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C$.

Since we have shown that each side of Equation (**??**) is a subset of the other, we have proven that if Λ is a non-empty indexing set and $\mathscr{A} = \{A_\alpha | \alpha \in \Lambda\}$ is an indexed family of sets, then the sets in Equation (**??**) are equal to each other. □

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.