Math 239 01
Prof. Doug Baldwin

Problem Set 7 — Sets

Complete by **Wednesday, April 3**
Grade by **Friday, April 5**

Purpose

This problem set develops your ability to work with and reason about sets. It also gives you some practice applying what you know about proofs in contexts other than the typical ones for this course. In particular, when you finish this problem set you should be able to . . .

- Prove theorems about sets using the choose-an-element method
- Use other proof techniques to prove theorems about sets
- Solve problems involving common set operations and concepts
- Apply proof techniques to proofs about real numbers
- Write formal proofs.

Background

Much of this exercise draws on material from sections 5.1 and 5.2 of our textbook. We discussed that material in class on March 26 and 28. In addition, this exercise uses proof techniques from chapters 3 and 4 of the book.

Activity

Solve the following problems. All proofs must be written according to conventions for formal proofs, including typeface rules (e.g., italic variables, emphasized labels for theorems and proofs, etc.).

Question 1. Exercises 15a, b, and c in section 5.1 of our textbook: determine whether certain intervals of the real numbers are subsets of each other; find intersections, unions, and differences of intervals. See the textbook for details.

Solution: Part A. (a, b) is a proper subset of $(a, b]$ because every number in (a, b) is also in $(a, b]$, and additionally b is in $(a, b]$.

Part B. $[a, b]$ is not a subset of $(a, +\infty)$ because the former interval contains a but the latter doesn't.

Part Ci. $[-3, 7] \cap (5, 9] = (5, 7]$.

Part Cii. $[-3, 7] \cup (5, 9] = [-3, 9]$.

Part Ciii. $[-3, 7] - (5, 9] = [-3, 5]$.

Question 2. Exercise 17 in section 5.1 of our textbook: complete a proof of the claim that if set S has n elements, then there are 2^n different subsets of S. See the book for more information, including guidance on how to complete the proof. Note that because this exercise doesn't call for you to write a complete proof, your solution needn't be a formal proof.

Solution: Part A. $P(0)$ is "if a set has exactly 0 elements, then that set has exactly $2^0 = 1$ subsets." A set with 0 elements is the empty set, whose only subset is the empty set itself. Thus the set with 0 elements has 1 subset.

Part B. $P(1)$ is "if a set has exactly 1 element, then that set has exactly $2^1 = 2$ subsets." Without loss of generality, let the set be $\{a\}$. Then its subsets are $\emptyset$ and $\{a\}$, so it has 2 subsets. Similarly, $P(2)$ is "if a set has exactly 2 elements, then that set has exactly $2^2 = 4$ subsets," and letting the set be $\{a, b\}$, its possible subsets are $\emptyset$, $\{a\}$, $\{b\}$, and $\{a, b\}$, a total of 4 subsets.

Part C. As outlined in the text, we assume that any set with exactly $k \geq 0$ elements has exactly 2^k subsets, and we let T be a set with $k + 1$ elements. Let x be any element of T, and define B to be $T - \{x\}$. Now we have that $T = B \cup \{x\}$, and so by Lemma 5.6 the subsets of T are the subsets of B, and the sets of the form $C \cup \{x\}$ where C is a subset of B. Note that there is one set of this form for each subset of B, and none of those sets is a subset of B, because B does not contain x. Thus T has twice as many subsets as B does. Since B contains k elements, the induction hypothesis applies to it, so B has 2^k subsets. Twice this is $2 \times 2^k = 2^{k+1}$.

Question 3. Exercise 12b in section 5.2 of our textbook: prove that if A, B, and C are subsets of some universal set, and $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Solution:

Theorem 1. If A, B, and C are subsets of some universal set, and $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Proof. We assume that A, B, and C are subsets of some universal set, and $A \subseteq B$, and will show that $A \cup C \subseteq B \cup C$. Let x be an element of $A \cup C$. Then $x \in A$ or $x \in C$. We consider each case separately.

If $x \in A$, then because $A \subseteq B$, x is also in B. Therefore $x \in B \cup C$.

If $x \in C$, then it follows immediately that $x \in B \cup C$.

Since we have shown that in both cases, $x \in A \cup C$ means that x is also in $B \cup C$, we have proven that $A \cup C \subseteq B \cup C$. □

Question 4. (This is an example of what I call a "Proofs Out of Context" problem, i.e., a problem that asks you to apply proof techniques from this course in contexts that you haven't necessarily seen before. This particular problem is adapted from a set of out of context problems for Math 239 developed by Prof. Olympia Nicodemi.)

Assume the following is true:

Fact 1. For all real numbers x and y, if $x > 0$ then there is a natural number n such that $nx > y$.

Use Fact 1 to prove the following two claims:

Proposition 1. For all real numbers x, if $x > 0$ then there is a natural number n such that $0 < \frac{1}{n} < x$.

Corollary 1. For all real numbers x and y, if $x < y$ then there is a natural number n such that $x < x + \frac{1}{n} < y$.

Be prepared during grading to say what significance for the real numbers you draw from Proposition 1 and Corollary 1.

Solution:

Proposition 1. For all real numbers x, if $x > 0$ then there is a natural number n such that $0 < \frac{1}{n} < x$.

Proof. We assume that x is a real number greater than 0, and will show that there is a natural number n such that $0 < \frac{1}{n} < x$. Since x and 1 are both real numbers, and $x > 0$, it follows from Fact 1 that there is some natural number n such that

$$nx > 1$$

From this inequality and the fact that $0 < 1$ it follows that

$$0 < 1 < nx$$

Dividing each part of this inequality by n (which we can do, since n is a natural number and so not equal to 0) yields

$$0 < \frac{1}{n} < x$$

We have thus shown that for all real numbers x, if $x > 0$ then there is a natural number n such that $0 < \frac{1}{n} < x$. □

Corollary 1. For all real numbers x and y, if $x < y$ then there is a natural number n such that $x < x + \frac{1}{n} < y$.

Proof. We assume that x and y are real numbers such that $x < y$, and show that there is a natural number n such that $x < x + \frac{1}{n} < y$. Since $x < y$, $y - x > 0$. Thus from Proposition 1, there exists a natural number n such that

$$0 < \frac{1}{n} < y - x$$

Adding x to each part of this inequality yields

$$x < x + \frac{1}{n} < y$$

We have thus shown that for all real numbers x and y, if $x < y$ then there is a natural number n such that $x < x + \frac{1}{n} < y$. □

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.