Math 239 01
Prof. Doug Baldwin

Problem Set 5 — Proofs by Contradiction and Cases

Complete by **Tuesday, March 12**
Grade by **Friday, March 15**

Purpose

This problem set develops your ability to do proofs in cases, and proofs by contradiction. When you finish this problem set you should be able to ...

- Prove claims by contradiction
- Prove claims using cases
- Write formal proofs.

Background

Our textbook discusses proof by contradiction in section 3.3, and proofs in cases in section 3.4. We discussed proof by contradiction in class on February 27 and March 1, and proofs in cases on March 4.

Activity

Solve the following problems. All proofs must be written according to conventions for formal proofs, including typeface rules (e.g., italic variables, emphasized labels for theorems and proofs, etc.); pay particular attention to the relatively new guideline of stating the method that a proof uses early in that proof.

Question 1. The proposition implied by exercise 8b in section 3.3 of our textbook. You will need to start by formulating the proposition, assuming that the special case in part a is true. (The proposition from part a is "for all real numbers x, $x + \sqrt{2}$ is irrational, or $-x + \sqrt{2}$ is irrational"; see the textbook for more information).

Solution:

Proposition 1. For all real numbers x and irrational numbers y, $x + y$ is irrational or $-x + y$ is irrational.

Proof. We assume that x is a real number and that y is an irrational number, and prove by contradiction that $x + y$ is irrational or $-x + y$ is irrational. Suppose for the sake of contradiction that $x + y$ and $-x + y$ are both rational. Then, because the rational numbers are closed under addition, the sum of these two expressions is also rational, i.e., $(x+y)+(-x+y)=2y$ is rational. Following the definition of the rationals, let $2y=\frac{a}{b}$ for integers a and b, $b \neq 0$. Then

$$\begin{aligned} \frac{2y}{2} &= \frac{a}{2b} \\ &= y \end{aligned}$$

is also rational, since $2b$ is an integer by closure, and not equal to 0 since $b \neq 0$. This contradicts the fact that y is irrational. We have thus shown that for all real numbers x and irrational numbers y, $x+y$ is irrational or $-x+y$ is irrational. □

Question 2. Exercise 16b in section 3.3 of Sundstrom's text (show that for all integers a and b, $b^2 \neq 4a+2$.)

Solution:

Proposition 2. For all integers a and b, $b^2 \neq 4a+2$.

Proof. We assume that a and b are integers, and show by contradiction that $b^2 \neq 4a+2$. Assume for the sake of contradiction that a and b are integers and $b^2 = 4a+2$. Since $4a+2$ can be written as $2(2a+1)$ and integers are closed under addition and multiplication, we see that b^2 is even. This in turn implies, by Sundstrom's Theorem 3.7, that b is even. We therefore have $b=2c$ for some integer c, and thus $b^2=4c^2$. But we also know that $b^2=4a+2$, so $4a+2=4c^2$. Dividing both sides of this equation by 4 yields $a+\frac{1}{2}=c^2$. Now a is an integer, and c^2 must be an integer because integers are closed under multiplication, so a and c^2 are two integers that differ by $\frac{1}{2}$, which is impossible. We have thus reached a contradiction, and so conclude that for all integers a and b, $b^2 \neq 4a+2$. □

Question 3. Exercise 5a in section 3.4 of Sundstrom's text (prove that for all integers a, b, and d with $d \neq 0$, if d divides a or d divides b, then d divides ab).

Solution:

Proposition 3. For all integers a, b, and d with $d \neq 0$, if d divides a or d divides b, then d divides ab.

Proof. We assume that a, b, and d are integers with $d \neq 0$, and show that if d divides a or d divides b, then d divides ab. The proof is in two cases, as follows:

Case 1: d divides a. Since d divides a, we can write a as $a=kd$ for some integer k. Then $ab=(kd)b=(kb)d$. Since integers are closed under multiplication, kb is an integer, and so d divides ab.

Case 2: d divides b. Similar to the first case, $b=kd$ for some integer k and $ab=a(kd)=(ka)d$ and so d divides ab.

We have now shown that in all cases allowed by the proposition, d divides ab, and so have shown that for all integers a, b, and d with $d \neq 0$, if d divides a or d divides b, then d divides ab. □

Question 4. Exercise 7 in section 3.4 of Sundstrom's text (determine whether it is true or false that for all integers n, if n is odd then $8|(n^2-1)$; prove the proposition or provide a counterexample, according to whether you think it is true or false). Recall that the notation $a|b$ means "a divides b," i.e., $b=ka$ for some integer k.

Solution: The claim is true:

Proposition 4. For all integers n, if n is odd then $8|(n^2-1)$.

Proof. We assume that n is an odd integer, and show that $8|(n^2-1)$. Since n is odd, we can write it as $n = 2a+1$ for some integer a. The proof then proceeds by cases, according to whether a is odd or even.

In the first case, a is odd, i.e., $a = 2b+1$ for some integer b. Thus

$$\begin{array}{rcl} n^2-1 & = & (2(2b+1)+1)^2-1 \\ & = & (4b+3)^2-1 \\ & = & (16b^2+24b+9)-1 \\ & = & 16b^2+24b+8 \\ & = & 8(2b^2+3b+1) \end{array}$$

Since integers are closed under multiplication and addition, $2b^2+3b+1$ is an integer, and so we see that 8 divides n^2-1.

In the second case, a is even, i.e., $a = 2b$ for some integer b. Then

$$\begin{array}{rcl} n^2-1 & = & (2(2b)+1)^2-1 \\ & = & (4b+1)^2-1 \\ & = & 16b^2+8b+1-1 \\ & = & 16b^2+8b \\ & = & 8(2b^2+b) \end{array}$$

As in the first case, $2b^2+b$ is an integer by closure, and so we see that 8 divides n^2-1.

We have now shown that in all cases, 8 divides n^2-1, and so we conclude that for all integers n, if n is odd then $8|(n^2-1)$. □

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.