Math 239 01
Prof. Doug Baldwin

Problem Set 4 — Methods of Proof

Complete by **Sunday, March 3**
Grade by **Tuesday, March 5**

Purpose

This problem set develops your ability to reason with contrapositives of conditionals, and to reason about biconditionals. It also gives you some practice working with set builder notation and quantifiers. When you finish this problem set you should be able to ...

- Write set builder descriptions of sets
- Determine whether quantified statements are true or false
- Prove conditional statements by using their contrapositives
- Prove claims about biconditional statements
- Write formal proofs
- Evaluate whether formal proofs are valid or not.

Background

The proof techniques exercised in this problem set come from section 3.2 of our textbook. We discussed that material in class on February 20 and 25. Set builder notation is in section 2.3, and was covered in class on February 13, while quantifiers are in section 2.4 and were discussed on February 15.

Activity

Answer the following questions. All proofs must be printed according to conventions for formal proofs, including typeface rules (e.g., italic variables, emphasized labels for theorems and proofs, etc.)

Question 1. Use set builder notation to describe each of the following sets:

1. The set of integer multiples of 7.
2. The interval $[-3, 8)$ on the real number line.
3. The interval $[-3, 8)$ in the rationals.
4. The set of integers n such that $2n$ is either greater than 3 or less than -11.

Solution:

1. $\{7n|n \in \mathbb{Z}\}$
2. $\{x \in \mathbb{R}| -3 \leq x < 8\}$
3. $\{x \in \mathbb{Q}| -3 \leq x < 8\}$
4. $\{n \in \mathbb{Z}|2n < -11 \vee 2n > 3\}$

Question 2. Classify each of the following as true or false, and justify each answer. Recall that for purposes of this course, the natural numbers are the integers strictly greater than 0.

1. All natural numbers greater than 9 are also greater than 10.
2. Some natural number greater than 9 is also greater than 10.
3. For all natural numbers n, there exists some natural number m such that $m > n$.
4. There exists some natural number n such that for all natural numbers m, $m > n$.
5. All natural numbers less than 1 are multiples of 10.
6. Some natural number less than 1 is a multiple of 10.

Solution:

1. False. 10 is a counterexample.
2. True. 11 is an example.
3. True. $m = n + 1$ is a natural number and is greater than n.
4. False. No matter what n is, it isn't greater than itself.
5. Vacuously true.
6. False. There are no examples.

Question 3. Exercise 5 in section 3.2 of our textbook, giving a formal proof if you decide the statement is true. (Exercise 5 asks you to consider the statement that for all integers a and b, if ab is even then a is even or b is even, and to either prove the statement true or give a counter-example to show that it is false.)

Solution: The statement is true, and can be proven as follows:

Theorem 1. For all integers a and b, if ab is even then a is even or b is even.

Proof. We assume that a and b are integers, and show by proving the contrapositive that if ab is even then a is even or b is even. The contrapositive of this statement is that if a is odd and b is odd, then ab is odd. This follows immediately from Theorem 1.8 in the textbook. This establishes the contrapositive, and so we have shown that for all integers a and b, if ab is even then a is even or b is even. □

Question 4. Prove that for all integers n, n is divisible by 6 if and only if n is divisible by 2 and n is divisible by 3. (Hint: the proposition explored in Question 3 might be helpful somewhere in your proof.)

Solution:

Theorem 2. For all integers n, n is divisible by 6 if and only if n is divisible by 2 and n is divisible by 3.

Proof. We assume that n is an integer, and show that n is divisible by 6 if and only if it is divisible by 2 and by 3. We prove each direction separately.

If: we assume that n is divisible by 2 and by 3, and show that n is divisible by 6. Since n is divisible by 2 and by 3, we can write n as

$$n = 2a \tag{1}$$

and

$$n = 3b \tag{2}$$

for some integers a and b. From equation 1, we see that n must be even. But then since 3 is odd in equation 2, the proposition from question 3 requires that b be even, i.e., $b = 2c$ for some integer c. Now we can write n as

$$\begin{aligned} n &= 3(2c) \\ &= 6c \end{aligned}$$

Thus, since c is an integer, we see that n is divisible by 6.

Only if: we assume that n is divisible by 6 and show that it is divisible by 2 and by 3. Since n is divisible by 6, there is some integer a such that

$$\begin{aligned} n &= 6a \\ &= 2(3a) \quad (3) \\ &= 3(2a) \quad (4) \end{aligned}$$

Since integers are closed under multiplication, $3a$ and $2a$ are both integers. Thus equation 3 shows that n is divisible by 2, and equation 4 that n is divisible by 3. Thus n is divisble by 2 and by 3.

We have now proven both directions of the biconditional, and so have shown that for all integers n, n is divisible by 6 if and only if n is divisible by 2 and n is divisible by 3. □

Question 5. Exercise 19a in section 3.2 of our textbook. This is an "evaluation of proofs" exercise, i.e., one that presents you with a proposition and a candidate proof of it, and asks you to determine whether the proposition is true and the proof valid, correcting invalid proofs. See the textbook for complete instructions, and the specific proposition and proof.

Solution: The proposition is

Theorem 3. If m is an odd integer, then $m + 6$ is an odd integer.

This proposition is true, but the proof is backwards, i.e., it assumes the conclusion of the proposition and proves the hypothesis. A corrected proof could be:

Proof. We assume that m is an odd integer, and show that $m + 6$ is an odd integer. Since m is odd, we can write it as

$$m = 2a + 1$$

for some integer a. Adding 6 to both sides of this equation produces

$$\begin{aligned} m+6 &= 2a+1+6 \\ &= 2(a+3)+1 \end{aligned}$$

Since the integers are closed under addition, $a+3$ is an integer, and so $m+6$ is an odd integer. We have thus shown that if m is an odd integer, then $m+6$ is an odd integer. $\square$

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.