Math 239 01
Prof. Doug Baldwin

Problem Set 2 — Direct Proofs

Complete by **Monday, February 11**
Grade by **Wednesday, February 13**

Purpose

This problem set develops your ability to write direct proofs. It also continues developing your ability to reason about logical propositions. When you finish this problem set you should be able to ...

- Give logically sound direct arguments for why claims involving numbers and their algebra are true
- Write formal direct proofs
- Draw correct inferences from compound propositions
- Read and write compound propositions using symbolic logical notation
- Construct truth tables for compound propositions
- Use truth tables to recognize tautologies and/or contradictions.

Background

This problem set is based on sections 1.2 and 2.1 of our textbook. We discussed section 1.2 in class on January 30, and will discuss 2.1 on February 4. You may also find the introduction to LaTeXfrom the February 1 class helpful for this problem set.

Activity

Answer the following questions. All proofs must be printed according to conventions for formal proofs, including typeface rules (e.g., italic variables, emphasized labels for theorems and proofs, etc.)

Question 1. Prove that if n is an even integer, then $4n + 2$ is an even integer.

Solution:

Theorem 1. If n is an even integer, then $4n + 2$ is an even integer.

Proof. We assume that n is an even integer, and will show that $4n+2$ is an even integer. Since n is even, n can be written as $n = 2a$ for some integer a. Then

$$\begin{aligned} 4n+2 &= 4(2a)+2 \\ &= 2(4a)+2 \\ &= 2(4a+1) \end{aligned}$$

Since the integers are closed under multiplication and addition, $4a+1$ is an integer, and so we have shown that $4n+2$ is of the form two times some integer, and thus that $4n+2$ is even (and an integer, by closure of the integers under multiplication). □

Question 2. (Exercise 11b in our textbook). Prove that the product of the two solutions to the quadratic equation is $\frac{c}{a}$. See the textbook for more discussion of this problem, including definitions of the variables.

Solution:

Theorem 2. If x_1 and x_2 are solutions to the equation $ax^2 + bx + c = 0$ where a, b, and c are real numbers with $a \neq 0$, then $x_1 x_2 = \frac{c}{a}$.

Proof. We assume that x_1 and x_2 are solutions to the equation $ax^2 + bx + c = 0$ where a, b, and c are real numbers with $a \neq 0$, and will show that $x_1 x_2 = \frac{c}{a}$. From the quadratic equation we know without loss of generality that

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Thus

$$\begin{aligned} x_1 x_2 &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \\ &= \frac{b^2 - (b^2 - 4ac)}{4a^2} \\ &= \frac{4ac}{4a^2} \\ &= \frac{c}{a} \end{aligned}$$

We have thus shown that if x_1 and x_2 are solutions to the equation $ax^2 + bx + c = 0$ where a, b, and c are real numbers with $a \neq 0$, then $x_1 x_2 = \frac{c}{a}$. □

Question 3. (An extension of exercise 8b in section 2.1 of our textbook.) Assume that the statements (1) "Laura is in the seventh grade," (2) "Laura got an A on the mathematics test or Sarah got an A on the mathematics test," and (3) "If Sarah got an A on the mathematics test then Laura is not in the seventh grade" are all true.

Part A. Determine whether the statement "Sarah got an A on the mathematics test" is true or false. You do not have to write a formal proof.

Solution: Since we know that Laura is in the seventh grade (statement 1), the only way statement 3 can be true is if Sarah did not get an A on the test. Then the only way statement 2 can be true if if Laura did get an A.

Part B. Letting G stand for "Laura is in the seventh grade," L stand for "Laura got an A on the mathematics test," and S stand for "Sarah got an A on the mathematics test," rewrite the 3-part assumption given to you in the introduction to this question as a single logical expression involving G, L, and S and various logical connectives (with parentheses as needed).

Solution: The individual statements are

1. "Laura is in the seventh grade": G
2. "Laura got an A on the mathematics test or Sarah got an A on the mathematics test": $L \vee S$
3. "If Sarah got an A on the mathematics test then Laura is not in the seventh grade": $S \to \neg G$

Combine these with "and" to get the complete assumption:

$$G \wedge (L \vee S) \wedge (S \to \neg G)$$

Question 4. Let P and Q be propositions. For each of the following statements, use truth tables to determine whether the statement is a tautology, a contradiction, or neither.

Part A. $(P \to Q) \to (\neg Q \to \neg P)$.

Solution:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$	$(P \to Q) \to (\neg Q \to \neg P)$
F	F	T	T	T	T	T
F	T	T	F	T	T	T
T	F	F	T	F	F	T
T	T	F	F	T	T	T

Part B. $(P \to Q) \wedge (P \to \neg Q)$.

Solution:

P	Q	$\neg Q$	$P \to Q$	$P \to \neg Q$	$(P \to Q) \wedge (P \to \neg Q)$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	T	F
T	T	F	T	F	F

Part C. $(P \to Q) \wedge P \wedge \neg Q$.

Solution:

P	Q	$\neg Q$	$P \to Q$	$(P \to Q) \wedge P \wedge \neg Q$
F	F	T	T	F
F	T	F	T	F
T	F	T	F	F
T	T	F	T	F

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.