Theorem 1. For all integers $d \geq 2$, if n is an integer such that d divides n, then d does not divide $n+1$.

Proof. We will show that for all integers $d \geq 2$, if n is an integer such that d divides n, then d does not divide $n+1$ by contradiction. Assume for the sake of contradiction that n is an integer such that some integer $d \geq 2$ divides n and also divides $n+1$. Since d divides n and $n+1$, we have

$$n = kd \tag{1}$$

and

$$n + 1 = cd \tag{2}$$

for some integers k and c. Adding 1 to both sides of Equation 1 gives

$$\begin{aligned} kd + 1 &= n + 1 \\ &= cd \end{aligned}$$

where the last line is from Equation 2. Dividing by d we see that

$$k + \frac{1}{d} = c$$

But since $d \geq 2$, $\frac{1}{d} \leq \frac{1}{2}$ and so it is impossible for $k + \frac{1}{d}$ to be an integer. This is a contradiction, and so we have proven by contradiction that for all integers $d \geq 2$, if n is an integer such that d divides n, then d does not divide $n+1$. $\square$

Theorem 2. There are an infinite number of prime numbers.

Proof. We prove that there are an infinite number of prime numbers by contradiction. Assume for the sake of contradiction that there are a finite number of primes. Then we can list the primes as $\{p_1, p_2, \ldots, p_n\}$ for some finite natural number n. Now consider the product $P = p_1 p_2 \ldots p_n$. Since all primes are greater than or equal to 2, Theorem 1 implies that none of p_1 through p_n divide $P+1$. This means that $P+1$ is not a product of prime numbers, and since every integer greater than or equal to 2 is either prime or a product of primes, $P+1$ must be a prime number. Further, $P+1$ is bigger than every member of the set $\{p_1, p_2, \ldots, p_n\}$, and so is not in that set. But this contradicts the assumption that the only primes are the numbers $\{p_1, p_2, \ldots, p_n\}$. We have thus shown by contradiction that there must be an infinite number of prime numbers. $\square$

Comment: note that an equivalent statement to this theorem is that there is no largest prime number.