Theorem 1. For all natural numbers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

Proof. We prove by induction on n that Equation 1 holds for all natural numbers n.

For the basis step we show that

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

This follows because

$$\begin{aligned}\sum_{i=1}^{1} i &= 1 \\ &= \frac{1 \times 2}{2} \\ &= \frac{1(1+1)}{2}\end{aligned}$$

Turning to the induction step, we must show that whenever Equation 1 holds for $n = k$, it also holds for $n = k + 1$. We therefore assume that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

and will show that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

We do this by rewriting the left-hand side:

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

Having shown that Equation 1 holds when $n = 1$, and that if Equation 1 holds for $n = k$ it also holds for $n = k + 1$, we have proven by induction that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for all natural numbers n. □