Math 239 — Sample Questions for Hour Exam 2

Spring, 2018

This document is a collection of questions relevant to our upcoming exam that I have used on past Proofs exams. I've included the original point value of each question, as an indication of how "big" I think each is (our exam will have a total of 50 points). All of the questions address material that might appear on our exam, but there are more questions here than will appear on it. I've included my solutions to each question, but I strongly recommend that you try to answer each question for yourself before looking at the solutions. My proofs italicize variable names, since I am typing my solutions, whereas you can write yours by hand and will not need to distinguish italic from regular characters.

Question 1 (15 Points). Prove that the number $5 - \sqrt{2}$ is irrational. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution: We prove by contradiction that $5 - \sqrt{2}$ is irrational. Assume for the sake of contradiction that $5 - \sqrt{2}$ is rational, i.e., there exist integers a and b, with $b \neq 0$, such that $5 - \sqrt{2} = a/b$. Algebraically rearranging this equation yields $\sqrt{2} = 5 - a/b$, which is rational since the rationals are closed under subtraction. But this is a contradiction, because we have previously proven that $\sqrt{2}$ is irrational. We have thus proven by contradiction that $5 - \sqrt{2}$ must be irrational. QED.

Question 2 (15 Points). Let the function $B(x)$ be defined as

$$B(x) = \begin{cases} x & \text{If } 0 \leq x < 1 \\ 2 - x & \text{If } 1 \leq x \leq 2 \\ 0 & \text{Otherwise} \end{cases}$$

($B(x)$ is a "first-degree basis spline." It's not very useful in its own right, but is a simple member of a family of functions that are widely used for approximating complicated curves with low-degree polynomials, among other things.)

Prove that for all real numbers x, $0 \leq B(x) \leq 1$. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution: We use cases to prove that for all real numbers x, $0 \leq B(x) \leq 1$. The cases correspond to the cases in the definition of $B(x)$, namely…

Case 1: $0 \leq x < 1$. In this case $B(x) = x$, and so $0 \leq B(x) < 1$ which in turn means $0 \leq B(x) \leq 1$.

Case 2: $1 \leq x \leq 2$. In this case $B(x) = 2 - x$. Since $1 \leq x \leq 2$, we have $2 - 1 \geq 2 - x \geq 2 - 2$, or $0 \leq 2 - x \leq 1$. Since $B(x) = 2 - x$, these last inequalities show that $0 \leq B(x) \leq 1$.

Case 3: $x < 0$ or $x > 2$. In this case $B(x)$ is defined by be 0, so $0 \leq B(x) \leq 1$.

These three cases cover all possible real values for x, and so together prove that for all real numbers x, $0 \leq B(x) \leq 1$. QED.

Question 3 (15 Points). Prove that if a and b are integers and the product ab is not divisible by 9, then at least one of a and b is not divisible by 3.

Solution: We prove that if a and b are integers and the product ab is not divisible by 9, then at least one of a and b is not divisible by 3 by proving the contrapositive. In other words, we prove that if integers a and b are both divisible by 3, then ab is divisible by 9. Assume that a and b are integers divisible by 3, i.e., that there exist integers x and y such that $a = 3x$ and $b = 3y$. We can then write the product ab as

$$
\begin{aligned}
ab &= (3x)(3y) \\
&= 9xy
\end{aligned}
$$

Since the integers are closed under multiplication, xy is an integer and so ab is a multiple of 9. We have thus proven the contrapositive, and so have also proven that if a and b are integers and the product ab is not divisible by 9, then at least one of a and b is not divisible by 3. QED.

Question 4 (20 Points). Prove that for all natural numbers n, 11^n is odd.

Solution: We use induction on n to prove that for all natural numbers n, 11^n is odd.

For the base case, consider $n = 1$. $11^1 = 11$, which is odd.

For the induction step, we show that if k is a natural number such that 11^k is odd, then 11^{k+1} is also odd. Assume that that 11^k is odd, and note that $11^{k+1} = 11 \times 11^k$. We know that both 11 and 11^k are odd, and that the product of two odd numbers is odd. Thus we conclude that 11^{k+1} is odd.

We have now established both the base case and the induction step, and so have shown by mathematical induction that for all natural numbers n, 11^n is odd. QED.

Question 5 (15 Points). Prove that no integer n has the property that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$.

Solution: We prove by contradiction that no integer n has the property that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$. Assume for the sake of contradiction that n is an integer and that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$. Since $n \equiv 1 \pmod 2$, we know that there is some integer x such that $n = 2x + 1$. Similarly, there is some integer y such that $n = 6y + 4$. Since these expressions both equal n, we have an equality that we can rearrange as follows:

$$2x + 1 = 6y + 4$$

$$2x - 6y = 3$$

$$2(x - 3y) = 3$$

Since the integers are closed under multiplication and subtraction, $x - 3y$ is an integer, and so we have shown that 3 is even. But this is a contradiction, since 3 is clearly odd. We thus conclude that no integer n has the property that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$. QED.

Question 6 (5 Points). Are the following statements logically equivalent? Explain why or why not in a sentence or two.

Statement 1: There is no mathematician who likes every theorem.

Statement 2: Every mathematician has some theorem that they don't like.

Solution: Think of the statements as quantified statements: statement 1 is "it is not the case that there exists a mathematician, M, such that for every theorem, T, M likes T." Applying rules for negating quantified statements, this becomes

"For all mathematicians, M, it is not the case that for all theorems, T, M likes T"

"For all mathematicians, M, there exists a theorem, T, such that M does not like T."

This last is Statement 2, so the original statements are equivalent.

Question 7 (15 Points). Prove that for all integers a and b, with $b \neq 0$, the rational number a/b is an integer if and only if b divides a. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution: We assume that a and b are integers, with $b \neq 0$, and show that a/b is an integer if and only if b divides a. We prove each direction of the biconditional separately.

For the first direction, we assume that b divides a, and show that a/b is an integer. Since b divides a, there is some integer k such that $a = kb$. Therefore $a/b = kb/b = k$, which is an integer.

For the second direction, we assume that a/b is an integer, call it m, and show that b divides a. From $a/b = m$, we have $a = mb$, showing that a is an integer multiple of b, i.e., b divides a.

Since we have shown both that if b divides a then a/b is an integer, and that if a/b is an integer then b divides a, we have proven that for all integers a and b, with $b \neq 0$, the rational number a/b is an integer if and only if b divides a. QED.

Question 8 (15 Points). Prove that for all real numbers *x*, if *x* is irrational then $\sqrt[3]{x}$ is also irrational. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution. We prove by contradiction that for all real numbers *x*, if *x* is irrational then $\sqrt[3]{x}$ is also irrational. So assume that *x* is an irrational number but $\sqrt[3]{x}$ is rational, i.e.,

$$\sqrt[3]{x} = \frac{a}{b}$$

for some integers *a* and *b* with $b \neq 0$. Then *x* must equal a^3/b^3, which is rational because the integers are closed under multiplication, and b^3 is not 0. But this is a contradiction, because a number cannot be both irrational and rational. Since we have shown that it is impossible for $\sqrt[3]{x}$ to be rational, we have proven that for all real numbers *x*, if *x* is irrational then $\sqrt[3]{x}$ is also irrational. QED.

Question 9 (15 Points). Prove that for all integers n, if $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ then n is even. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution. We assume that $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ and prove that n must be even. The proof is in cases, corresponding to $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$.

For the first case, suppose $n \equiv 0 \pmod 4$. Then $n = 4a + 0$ for some integer a, which is equivalent to $n = 2(2a)$. By the definition of "even," $2(2a)$ is an even number.

For the second case, suppose $n \equiv 2 \pmod 4$, which means $n = 4b + 2$ for some integer b. Rewriting, we have $n = 4b + 2 = 2(2b+1)$. Because integers are closed under addition and multiplication, $2b+1$ is an integer, and so once again n is even.

Having shown that n is even both when $n \equiv 0 \pmod 4$ and when $n \equiv 2 \pmod 4$, we have proven that for all integers n, if $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ then n is even. QED.