Math 239 — Sample Final Exam Questions

May 2019

General Comments. These are some questions I have used recently on final exams for this course. I show the point value associated with each question so you have some sense of how hard or important I considered each. My final exams for this course have 120 points total. However, since I have taken these questions from several exams, they total to more than 120 points.

Question 1. (20 points) Let A, B, and C be nonempty sets. Furthermore, let $f : A \to B$ and $g : B \to C$ be functions. Write a formal proof that if f and g are both injections, then $g \circ f$ is an injection.

Solution:

Proof. We prove that if A, B, and C are nonempty sets, and $f : A \to B$ and $g : B \to C$ are injections, then $g \circ f$ is an injection by showing that for all elements x and y of A, if $(g \circ f)(x) = (g \circ f)(y)$, then $x = y$. So assume that $(g \circ f)(x) = (g \circ f)(y)$, i.e., that $g(f(x)) = g(f(y))$. Since g is an injection, we must have $f(x) = f(y)$. And now, since f is also an injection, x must equal y. We have thus proved that if A, B, and C are nonempty sets, and $f : A \to B$ and $g : B \to C$ are injections, then $g \circ f$ is an injection. $\square$

Question 2. (20 points) Define the Math 239 Numbers to be the members of the set

$$\left\{\frac{n-239}{n} | n \in \mathbb{N}\right\}$$

Determine whether the set of all Math 239 Numbers is finite, countably infinite, or uncountably infinite, and support your answer with a formal proof.

Solution:

Theorem 1. The set of Math 239 numbers is countably infinite.

Proof. We show that the set of Math 239 numbers is countably infinite by showing a bijection between it and the natural numbers. In particular, the function

$$f(n) = \frac{n-239}{n}$$

is a bijection from the natural numbers to the Math 239 numbers, and so f^{-1} is a bijection from the Math 239 numbers to the naturals. To see that f is an injection, suppose that $f(n) = f(m)$ for natural numbers n and m. Then

$$\frac{n-239}{n} = \frac{m-239}{m}$$

and so

$$mn - 239m = mn - 239n$$

or

$$239m = 239n$$

and so $m = n$. To see that f is a surjection, note that f implements the definition of the Math 239 numbers, and so every Math 239 number is necessarily the image of some natural under f. We have thus shown that there is a bijection between the set of Math 239 numbers and the natural numbers, and so the set of Math 239 numbers is countably infinite. □

Question 3. (15 points) Define two points in the plane to be "about the same" if the distance between them is small. More precisely, define "about the same" to be a relation on ordered pairs of reals such that (x_1, y_1) is about the same as (x_2, y_2) if and only if

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \leq 1$$

Is "about the same" an equivalence relation? If so, give a formal proof; if not identify which properties of an equivalence relation it has and which it does not have, explaining but not formally proving each claim in a sentence or two.

Solution: "About the same" is reflexive and symmetric, but not transitive, and so it is not an equivalence relation.

"About the same" is reflexive because for every point (x, y),

$$\sqrt{(x - x)^2 + (y - y)^2} = 0 \leq 1$$

and so (x, y) is about the same as itself.

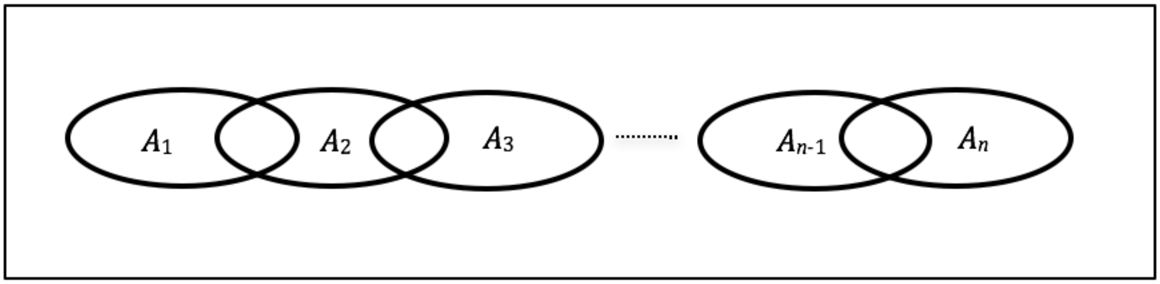
"About the same" is also symmetric, because for every pair of points (x_1, y_1) and (x_2, y_2),

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and so (x_1, y_1) is about the same as (x_2, y_2) if and only if (x_2, y_2) is about the same as (x_1, y_1).

However, "about the same" is not transitive because, for example $(0, 0)$ is about the same as $(0, 1)$, and $(0, 1)$ is about the same as $(0, 2)$, but $(0, 0)$ is not about the same as $(0, 2)$.

Question 4. (15 points) Let $n \geq 3$ be a natural number and let $\mathcal{A} = \{A_i | 1 \leq i \leq n\}$ be a family of sets. Informally, let each set in $\mathcal{A}$ have a non-empty intersection with the next and previous members of $\mathcal{A}$, but no other intersections, as in this Venn diagram:



Formally, for all natural numbers i such that $1 \leq i < n$, $A_i \cap A_{i+1} \neq \emptyset$, but for all natural numbers j and k such that $1 \leq j < k - 1 < n$, $A_j \cap A_k = \emptyset$. Give a formal proof that under these conditions,

$$\bigcap_{1 \leq i \leq n} A_i = \emptyset$$

Solution:

Proof. We prove by contradiction that under the conditions given in the question, $\bigcap_{1 \leq i \leq n} A_i = \emptyset$. Assume for the sake of contradiction that the intersection is not empty. Then there is some element, call it x, that is a member of the intersection, and hence of every member of $\mathcal{A}$. So in particular, $x \in A_1$ and $x \in A_3$, contradicting the requirement that for all natural numbers j and k such that $1 \leq j < k - 1 < n$, $A_j \cap A_k = \emptyset$ (specifically, the contradiction happens when $j = 1$ and $k = 3$, which is always possible since $n \geq 3$). We have thus shown that it is impossible for $\bigcap_{1 \leq i \leq n} A_i$ to not be empty, and therefore that $\bigcap_{1 \leq i \leq n} A_i = \emptyset$. $\square$

Question 5. (15 points) Let $f : \mathbb{Z}^+ \to \mathbb{N}$ (where $\mathbb{Z}^+$ denotes the non-negative integers) be defined piecewise as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f(n-1)^5 & \text{if } n > 0 \end{cases}$$

Write a formal proof that for all non-negative integers n, $f(n) = 1$.

Solution:

Proof. We prove by induction on n that $f(n) = 1$ for all non-negative integers n.

For the basis step, let $n = 0$. Then from the definition of f, $f(n) = f(0) = 1$.

For the induction step, we show that if $f(k) = 1$ for some non-negative integer k, then $f(k+1) = 1$. So assume that k is a non-negative integer such that $f(k) = 1$, and consider $f(k+1)$. Since k is a non-negative integer, $k + 1 > 0$, and so $f(k+1)$ is defined as

$$\begin{aligned} f(k+1) &= f((k+1)-1)^5 \\ &= f(k)^5 \\ &= 1^5 \\ &= 1 \end{aligned}$$

Since we have shown that $f(0) = 1$ and that whenever $f(k) = 1$, $f(k+1)$ also equals 1, it follows by the principle of mathematical induction that $f(n) = 1$ for all non-negative integers n. □

Question 6. (15 points) Consider the statement "every city has some neighborhood where everyone is rich, and some neighborhood where everyone is poor." This statement can be written in symbolic logical form as

$$(\forall c \in C)((\exists n \in N)(\forall x \in P)(L(x,n) \to R(x)) \wedge (\exists m \in N)(\forall y \in P)(L(y,m) \to P(y)))$$

Where

- C is a universal set of cities
- N is a universal set of neighborhoods
- P is a universal set of people
- $L(x,y)$ is a predicate meaning that person x lives in neighborhood y
- $R(x)$ is a predicate meaning that person x is rich
- $P(x)$ is a predicate meaning that person x is poor.

Derive the symbolic negation of the statement and paraphrase that negation in English. If you need the negation of some predicate, e.g., $Q(x)$, in your answer, just write "$\neg Q(x)$," don't invent a new predicate that means the opposite of Q.

Solution:

$$\begin{aligned}
& \neg(\forall c \in C)((\exists n \in N)(\forall x \in P)(L(x,n) \to R(x)) \wedge (\exists m \in N)(\forall y \in P)(L(y,m) \to P(y))) \\
\equiv\ & (\exists c \in C)\neg((\exists n \in N)(\forall x \in P)(L(x,n) \to R(x)) \wedge (\exists m \in N)(\forall y \in P)(L(y,m) \to P(y))) \\
\equiv\ & (\exists c \in C)(\neg(\exists n \in N)(\forall x \in P)(L(x,n) \to R(x)) \vee \neg(\exists m \in N)(\forall y \in P)(L(y,m) \to P(y))) \\
\equiv\ & (\exists c \in C)((\forall n \in N)(\neg(\forall x \in P)(L(x,n) \to R(x))) \vee (\forall m \in N)(\neg(\forall y \in P)(L(y,m) \to P(y)))) \\
\equiv\ & (\exists c \in C)((\forall n \in N)(\exists x \in P)(\neg(L(x,n) \to R(x))) \vee (\forall m \in N)(\exists y \in P)(\neg(L(y,m) \to P(y)))) \\
\equiv\ & (\exists c \in C)((\forall n \in N)(\exists x \in P)(L(x,n) \wedge \neg R(x)) \vee (\forall m \in N)(\exists y \in P)(L(y,m) \wedge \neg P(y)))
\end{aligned}$$

In English, this would be "in some city, either every neighborhood has a resident who is not rich, or every neighborhood has a resident who is not poor."

Question 7. (10 points) Let A and B be subsets of some universal set. Prove that

$$(A \cup B^C) \cap (B \cup A^C) = (A \cap B) \cup (A^C \cap B^C)$$

Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution:

Proof. We use set algebra to prove that if A and B are subsets of some universal set, then $(A \cup B^C) \cap (B \cup A^C) = (A \cap B) \cup (A^C \cap B^C)$. We begin by distributing the intersection on the left side of the equation over the unions:

$$\begin{aligned}
(A \cup B^C) \cap (B \cup A^C) &= (A \cap (B \cup A^C)) \cup (B^C \cap (B \cup A^C)) \\
&= (A \cap B) \cup (A \cap A^C) \cup (B^C \cap B) \cup (B^C \cap A^C) \\
&= (A \cap B) \cup \emptyset \cup \emptyset \cup (B^C \cap A^C) \\
&= (A \cap B) \cup (B^C \cap A^C) \\
&= (A \cap B) \cup (A^C \cap B^C)
\end{aligned}$$

This completes the algebraic proof that if A and B are subsets of some universal set, then $(A \cup B^C) \cap (B \cup A^C) = (A \cap B) \cup (A^C \cap B^C)$. □

Question 8. (10 points) Let m and n be even integers, and suppose that k is an integer such that $m \equiv k \pmod{n}$. Prove that k is even. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution:

Proof. We assume that m and n are even integers, and that k is an integer such that $m \equiv k \pmod{n}$, and will show that k must be even. From the definition of congruence modulo n, $m \equiv k \pmod{n}$ means that $m - k = cn$ for some integer c. Rearranging this equation, we see that

$$m - cn = k \tag{1}$$

Now since m and n are even, we can write $m = 2a$ and $n = 2b$ for some integers a and b. Substituting these expressions for m and n in Equation (1) yields

$$\begin{aligned} k &= m - cn \\ &= 2a - 2cb \\ &= 2(a - cb) \end{aligned}$$

Since the integers are closed under subtraction and multiplication, $a - cb$ is an integer, and so we have expressed k as an even integer. We have thereby shown that if m and n are even integers and k is an integer such that $m \equiv k \pmod{n}$, then k is even. □

Question 9. (15 points) In the most common modern system for describing dates, years are designated by a positive integer and either "CE" ("Common Era") or "BCE" ("Before the Common Era"). There is no year 0. So, for example, today would be described as being in the year 2019 CE, Euclid wrote his geometry book around the year 300 BCE, and after New Years Eve parties in the year 1 BCE people woke up in the year 1 CE. Prove that the set of possible year designations as described above is countably infinite. Assume for purposes of this proof that there is no earliest or latest possible year. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

Solution:

Proof. We show that the set of year designations described in the problem is countably infinite by showing a bijection from it to the integers. In particular, define function f to map year designations to integers by the following rules:

- For year y BCE, $f(y) = -y$
- For year y CE, $f(y) = y - 1$

This function is a bijection, because if $f(x) = f(y) = z$, then either z is negative or z is not negative. In the former case, x and y must both be year designations BCE, so $-x = -y$ and $x = y$. In the latter case x and y are both year designations CE, and so $x - 1 = y - 1$ and again $x = y$. Function f is a surjection because every negative integer is the image under f of a year BCE, and every non-negative integer z is the image of year $z + 1$ CE.

We have now shown that f is an injection and a surjection from year designations to integers, and thus that f is a bijection. This establishes that the set of year designations is equivalent to the set of integers. Since the integers are equivalent to the natural numbers, and equivalence is transitive, we have also established that the set of year designations is equivalent to the set of natural numbers, and thus that the set of year designations is countably infinite. Thus, by exhibiting a bijection from the set of year designations to the set of integers, we have proven that the set of year designations is countably infinite. □

Question 10. In one of my other classes, a recent problem set had a question about a space traveler collecting genesium, a rare and incredibly valuable substance without which galactic civilization could not exist. This was originally a calculus question, but it also has logic questions lurking inside it. In particular…

Part A. (5 points) Phrase the statement "without genesium, galactic civilization would not exist" as a conditional.

> **Solution:** "If there is no genesium, then galactic civilization does not exist," or similar.

Part B. (5 points) Assume that no galactic civilization exists. Do that assumption and your conditional from Part A together imply that there is no such thing as genesium? Explain why or why not in a sentence or two.

> **Solution:** No, because the non-existence of galactic civilization makes the conclusion to the conditional from Part A true, but with a true conclusion the overall conditional would be true with either a true or false hypothesis.

Part C. (5 points) Still assuming that no galactic civilizations exist, is the statement "all galactic civilizations use genesium" true or false. Explain why in a sentence or two.

> **Solution:** The statement is vacuously true.