Theorem 1. If A and B are finite subsets of some universal set, then $A \cup B$ is finite.

Proof. We assume that A and B are finite subsets of some universal set. Therefore there is a bijection $g : B \to \mathbb{N}_m$ for some natural number m. Furthermore, since $A - B \subseteq A$ and A is finite, so is $A - B$. Therefore there is a bijection $f : A - B \to \mathbb{N}_k$ for some natural number k. Now define $h : A \cup B \to \mathbb{N}_{k+m}$ by

$$h(x) = \begin{cases} f(x) + m \text{ if } x \in A - B \\ g(x) \text{ if } x \in B \end{cases}$$

To see that h is an injection, suppose $x \neq y$. Then if x and y are both elements of B or both elements of $A - B$, $h(x) \neq h(y)$ because g and f are injections. If, without loss of generality, $x \in B$ and $y \in A - B$, then $h(x) \neq h(y)$ because $h(x) \leq m$ while $h(y) > m$. To see that h is a surjection, note first that h uses g to map B to $\mathbb{N}_m$, and g is a surjection from B to $\mathbb{N}_m$. Second, f is a surjection from $A - B$ to $\mathbb{N}_k$, and for every $x \in A - B$, $h(x) = f(x) + m$, which is a surjection from $A - B$ to $\{1 + m, 2 + m, 3 + m, \ldots k + m\}$. Thus h maps some element of B to each element of $\{1, 2, 3, \ldots, m\}$, and some element of $A - B$ to each element of $\{1+m, 2+m, 3+m, \ldots, k+m\}$, so h is a surjection from $A \cup B$ to $\mathbb{N}_{k+m}$. Since h is an injection and a surjection, h is a bijection from $A \cup B$ to $\mathbb{N}_{k+m}$, establishing that $A \cup B \approx \mathbb{N}_{k+m}$, and so $A \cup B$ is finite. □