Theorem 1. Let F_n, $n \geq 0$, denote the n^{th} Fibonacci number. Then for all $n \geq 6$, $F_n \geq \sqrt{2}^n$.

Proof. We use strong induction on n to show that for all $n \geq 6$, $F_n \geq \sqrt{2}^n$.

For basis steps, we consider $n = 6$ and $n = 7$. When $n = 6$, we have $F_6 = 8 = \sqrt{2}^6$, and so the claim holds. For $n = 7$, we have $F_7 = 13$, and $\sqrt{2}^7 = \sqrt{2} \times 8$. Since $\sqrt{2} < 1.5$, $\sqrt{2} \times 8 < 12 \leq 13$, and so $F_7 \geq \sqrt{2}^7$.

For the induction step, we assume that for some natural number $k \geq 7$, $F_i \geq \sqrt{2}^i$ for all natural numbers $6 \leq i \leq k$. We then show that $F_{k+1} \geq \sqrt{2}^{k+1}$. We begin with the definition of F_{k+1}:

$$\begin{aligned} F_{k+1} &= F_{k-1} + F_k \\ &\geq F_{k-1} + F_{k-1} \\ &= 2F_{k-1} \\ &\geq 2\sqrt{2}^{k-1} \\ &= \sqrt{2}^{k+1} \end{aligned}$$

We have now shown that $F_6 \geq \sqrt{2}^6$, and $F_7 \geq \sqrt{2}^7$, and that for all $k \geq 7$, $F_i \geq \sqrt{2}^i$ for all natural numbers $6 \leq i \leq k$ implies that $F_{k+1} \geq \sqrt{2}^{k+1}$. It therefore follows by the strong principle of induction that for all $n \geq 6$, $F_n \geq \sqrt{2}^n$. □