Theorem 1. If $f(n)$ is defined for all natural numbers n by

$$f(n) = \begin{cases} 2 & \text{if } n = 1 \\ 2 + f(n-1) & \text{if } n > 1 \end{cases} \tag{1}$$

then $f(n) = 2n$.

Proof. We assume that $f(n)$ is defined by Equation 1, and prove that $f(n) = 2n$. The proof is by induction on n.

For the basis step, we let $n = 1$ and will show that $f(1) = 2$. From the first case in Equation 1, we see that $f(1)$ is in fact 2.

For the induction step, we need to show that if $f(k) = 2k$ for some natural $k \geq 1$, then $f(k+1) = 2(k+1)$. We assume $f(k) = 2k$, and calculate $f(k+1)$ as follows:

$$\begin{aligned} f(k+1) &= 2 + f(k) \\ &= 2 + 2k \\ &= 2(k+1) \end{aligned}$$

We have now shown that $f(1) = 2$ and that if $f(k) = 2k$ for some natural $k \geq 1$, then $f(k+1) = 2(k+1)$. Therefore, by the principle of mathematical induction, if $f(n)$ is defined for all natural numbers n by Equation 1, then $f(n) = 2n$. □