Theorem 1. If A, B, and C are sets, then $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Proof. We show that if A, B, and C are sets, then $(A \cap B) \times C = (A \times C) \cap (B \times C)$ by showing that each side of the equation describes a subset of the other.

To see that $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$, suppose that $u = (x, y)$ is an element of $(A \cap B) \times C$. Then from the definition of Cartesian product, we have $x \in A \cap B$ and $y \in C$. Since x is an element of both A and B, the pair (x, y) is in both $A \times C$ and $B \times C$; in other words $u = (x, y) \in (A \times C) \cap (B \times C)$. We have thus shown that $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

To see that $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$, suppose that $v = (x, y)$ is an element of $(A \times C) \cap (B \times C)$. This means that $x \in A$ and $y \in C$ and $x \in B$ and $y \in C$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Then the pair $v = (x, y) \in (A \cap B) \times C$, establishing that $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$.

Since we have now shown that $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$ and $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$, we have proven that if A, B, and C are sets, then $(A \cap B) \times C = (A \times C) \cap (B \times C)$. □